Asian Research Journal of Mathematics

16(10): 144-151, 2020; Article no.ARJOM.62489

ISSN: 2456-477X

Generalized Hadamard Matrices and 2 −Factorization of Complete Graphs

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Authors' contributions

This work was carried out in collaboration between both authors. Author WVN designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author AAIP managed the analyses of the study and the literature searches. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2020/v16i1030236
Editor(s):
(1) Dr. Junjie Chen, University of Texas at Arlington, USA.
Reviewers:
(1) Shuji Kawasaki, Iwate University, Japan.
(2) Julio Cezar Uzinski, Federal University Of Mato Grosso, Brazil.
Complete Peer review History: http://www.sdiarticle4.com/review-history/62489

Received: 22 September 2020
Accepted: 29 October 2020
Published: 27 November 2020

Original Research Article

Abstract

Graph factorization plays a major role in graph theory and it shares common ideas in important problems such as edge coloring and Hamiltonian cycles. A factor F of a graph G is a spanning subgraph of G which is not totally disconnected. An n- factor is an n- regular spanning subgraph of G and G is n-factorable if there are edge-disjoint n -factors $F_1, F_2, \dots, F_k$ such that $G = F_1 \cup F_2 \cup \dots \cup F_k$. We shall refer $\{F_1, F_2, \dots, F_k\}$ as an n-*factorization* of a graph G. In this research we consider 2-factorization of complete graph. A graph with n vertices is called a complete graph if every pair of distinct vertices is joined by an edge and it is denoted by K_n. We look into the possibility of factorizing K_n with added limitations coming in relation to the rows of generalized Hadamard matrix over a cyclic group. Over a cyclic group C_p of prime order p, a square matrix $H(p, v)$ of order v all of whose elements are the p^{th} root of unity is called a generalized Hadamard matrix if $HH^* = vI_v$, where H^* is the conjugate transpose of matrix H and I_v is the identity matrix of order v. In the present work, generalized Hadamard matrices $GH(3, 3^m)$ over a cyclic group C_3 have been considered. We prove that the factorization is possible for K_{3^m} in the case of the limitation 1, namely, If an edge $\{i, j\}$ belongs to the factor F_k, then the i^{th} and j^{th} entries of the corresponding generalized Hadamard matrix should be different in the k^{th} row. In Particular, $\frac{(n-1)}{2}$ number of rows in the generalized Hadamard matrices is used to form 2-factorization of complete graphs. We discuss some illustrative examples that might be used for studying the factorization of complete graphs.

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Keywords: Generalized Hadamard matrices; factor; factorization; Kronecker product; totally disconnected.

1 Introduction

There is an enormous body of work on factors and factorizations and this topic has much in common with other areas of study in graph theory [1]. For example, factorization significantly overlaps the concept of graph coloring (edge coloring). Moreover, the Hamilton cycle problem can be viewed as the search for a connected factor in which the degree of each vertex is exactly two [2].

Some of the fundamental definitions, notations and terminology which will be used in our work are given as follows. A graph G is said to be disconnected if there exists two vertices in G such that no path in G has those vertices as endpoints. A factor F of a graph G is a spanning subgraph of G which is not totally disconnected. The union of edge disjoint factors which forms G is called a factorization of a graph G. An $n-$factor is an $n-$regular spanning subgraph of G and G is $n-$factorable if there are edge-disjoint $n-$factors $F_1, F_2, \dots, F_k$ such that $G = F_1 \cup F_2 \cup \dots \cup F_k$. $\{F_1, F_2, \dots, F_k\}$ is referred as an $n-$factorization of a graph G [3]. The graph which admits $n-$factorization is called an $n-$factorable graph [4]. For example, if a factor F has all of its degrees equal to 2, it is called a $2-$factor and it leads to $2-$factorization.

In this research we consider $2-$factorization of complete graphs [5]. If every pair of distinct vertices are joined by an edge, we say that the graph $G = (V(G), E(G))$ is a complete graph and if, in addition, $|V(G)| = n$, the graph G is denoted by K_n [3].

If a graph is $2-$ factorable, then it has to be $2k-$ regular for some integer k. In [6], Julius Petersen showed that this necessary condition is also sufficient: any $2k-$ regular graph is $2-$ factorable. Thus, given a complete graph with odd number of vertices is $2-$factorable and number of factors r is $(n-1)/2$ for a complete graph with n vertices [5]. In Fig. 1, the number of vertices is 5, and thus number of $2-$ factors is $\frac{5-1}{2} = 2$.

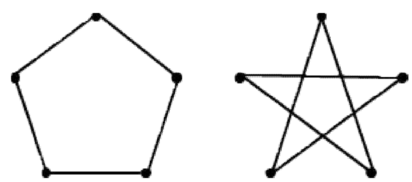

Fig. 1. $2-$ factorization of K_5

We look into the possibility of factorizing K_n with added limitations coming in relation to the rows of generalized Hadamard matrix over a cyclic group. Over a cyclic group C_p of prime order p, a square matrix $H(p, v)$ of order v all of whose elements are the p^{th} root of unity is called a generalized Hadamard matrix if $HH^* = vI_v$, where H^* is the conjugate transpose of matrix H and I_v is the identity matrix of order v.[7,8] $GH(2, v)$ matrices are referred to as classic Hadamard matrices. It is known that $GH(2, v)$ matrices can exist for $v = 1$,2 or $4k$, where k is a positive integer.

For primes $p > 2$, it has been conjectured that $GH(p, pt)$ exists for all positive integers t. In the present work, generalized Hadamard matrices $GH(3, 3^m)$ over a cyclic group C_3 have been considered these generalized Hadamard matrices were constructed in [9] since $p = 3$ is the smallest odd prime value. Note that, the elements of $GH(3, 3^m)$ are roots from the $\omega^3 = 1$.

The one the techniques we used in [9] was Kronecker product. Here also we define the term Kronecker product also known as the tensor product as it is very useful in this context. If $A = (a_{ij})$ is an $u \times v$ matrix for $i = 1,2, \dots, u$ and $j = 1,2, \dots, v$ and B is any $p \times q$ matrix then the Kronecker product of A and B, denoted by $A \otimes B$, is the $up \times vq$ matrix formed by multiplying each a_{ij} element by the entire matrix B. That is, [10]

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1v}B \\ a_{21}B & a_{21}B & \dots & a_{2v}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{u1}B & a_{u1}B & \dots & a_{uv}B \end{bmatrix}$$

Each generalized Hadamard matrix can be reduced by elementary operations (row and column commutation, their multiplication by a fixed root of unity) to a normalized generalized Hadamard matrix whose first row and first column consist of 1 [7,8,9,10].

We introduce the formal definition with the most familiar type of tournament, a complete Round Robin as it is useful for later constructions. A round robin tournament with an even number of teams, n, is a tournament of $n - 1$ rounds where each team plays the other $n - 1$ teams. A round is a collection of games where each team is matched with exactly one other team. This is often considered a fair tournament, and can be represented by a complete graph on n vertices. The vertices on the graph represent the teams, each labelled by its strength, and edges between vertices indicate the teams play each other in the tournament [11].

2 Materials and Methods

The normalized generalized Hadamard matrices are considered and the notations of the rows are started from the second row. Hence, H will be of the form [12]:

$$H = \begin{bmatrix} 1 & 1 & \dots & 1 \\ h_{11} & h_{12} & \dots & h_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{(n-1)1} & h_{(n-1)2} & \dots & h_{(n-1)n} \end{bmatrix}$$

For given a Hadamard matrix H, we want to find a $2-$ factorization $\left\{F_1, F_2, \dots F_k, \dots, F_{\left(\frac{n-1}{2}\right)}\right\}$ of K_n such that either each factor satisfies the limitations L_1 or L_2:

L_1: If an edge $\{i, j\}$ belongs to the factor F_k, then the i^{th} and j^{th} entries should be different in the k^{th} row:

$\{i, j\} \in F_k \Longrightarrow h_{ik} \neq h_{jk}$

L_2: If an edge $\{i, j\}$ belongs to the factor F_k, then the i^{th} and j^{th} entries should be same in the k^{th} row:

$\{i, j\} \in F_k \Longrightarrow h_{ik} = h_{jk}$

Note that,

If $\{i, j\} \in F_k$ then $\{i, n\} \in F_k$ and $\{m, j\} \in F_k$ for $n, m \neq i, j$.

The above condition should be satisfied in order to obtain the $2-$facorization. Consider the following example of Generalized Hadamard matrix of order 3 over C_3 to illustrate the factorization satisfying the limitation (L_1).

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

For the second row $(1, \omega, \omega^2)$, there are three edges $\{1,2\}, \{2,3\}, \{1,3\}$ have the potential to be selected. So, the $2-$ factor satisfying the limitation L_1 is $\{\{1,2\}, \{2,3\}, \{1,3\}\}$. For row $(1, \omega^2, \omega)$, we can do the same analysis and we get $\{\{1,2\}, \{2,3\}, \{1,3\}\}$ as $2-$ factor. Observe that $2-$ factors obtained from second and third rows are same. Thus, $2-$ factorization of complete graph on 3 vertices obtained from $GH(3,3)$ is $F_1 = \{\{1,2\}, \{2,3\}, \{1,3\}\}$. (Fig. 2)

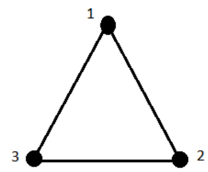

Fig. 2. $2-$ factorization of K_3 obtained from $GH(3,3)$

For the generalized Hadamard matrices $GH(3, 3^m)$ over a cyclic group C_p, the construction seems more complex than the simple example. For the general case, we consider the problem of finding the $2-$factorizations as follows:

Let $a^k_{\{i,j\}}$ is defined for every row k and pair of columns $\{i, j\}$ satisfying limitation L_1 by:

$$a^k_{\{i,j\}} = \begin{cases} 1 & if\ h_{ki} \neq h_{kj} \\ 0 & otherwise \end{cases} \tag{1}$$

and for each $k = 1,2, \dots, \frac{(n-1)}{2}$ and $i, j = 1,2, \dots, n$ such that $i \neq j$

$$\textstyle\sum_{i \neq j}^{n} a^k_{\{i,j\}} = 2 \tag{2}$$

and for each edge $\{i, j\}$ in the graph

$$\textstyle\sum_{k=1}^{\frac{(n-1)}{2}} a^k_{\{i,j\}} = 1 \tag{3}$$

Theorem 1: Let $m > 1$ be an integer and $n = 3^m$. Then there exist $2-$ factorization of K_n fulfilling the limitations L_1.

Proof.

The following proof is made by using Mathematical Induction. When $m = 1$ for generalized Hadamard matrix of order 3; H_1. It has been already shown that there is one and only one possible choice for a factorization satisfying L_1. That is $\{F_1\} = \{\{1,2\}, \{2,3\}, \{1,3\}\}$.

Inductive hypothesis states that there exist such factorizations for K_{3^m} satisfying the limitations.

Then we have

$$a^k_{\{i,j\}} = \begin{cases} 1 & if\ h_{ki} \neq h_{kj} \\ 0 & otherwise \end{cases} \tag{4}$$

and for each $k = 1,2, \dots, \frac{(3^m-1)}{2}$ and $i, j = 1,2, \dots, n$ such that $i \neq j$

$$\sum_{i \neq j}^{3^m} a_{\{i,j\}}^k = 2 \tag{5}$$

and for each edge $\{i, j\}$ in the graph

$$\sum_{k=1}^{\frac{(3^m-1)}{2}} a_{\{i,j\}}^k = 1 \tag{6}$$

Now, using the kronecker product

$$H_{m+1} = \begin{bmatrix} H_m & H_m & H_m \\ H_m & \omega H_m & \omega^2 H_m \\ H_m & \omega^2 H_m & \omega H_m \end{bmatrix}$$

In the $3^{m-1^{th}}$ row, only different things are $a_{\{3^m+1,3^m+2\}}^k, a_{\{3^m+1,3^m+3\}}^k, a_{\{3^m+2,3^m+3\}}^k$. This implies $\sum_{i \neq j}^{3^{m+1}} a_{\{i,j\}}^k = 2$.

If $\{i, j\}$ are adjacent in $3^{m-1^{th}}$ row, then $\{i, j\}$ are non-adjacent in any other rows. This implies $\sum_{k=1}^{\frac{(3^{m+1}-1)}{2}} a_{\{i,j\}}^k = 1$. This gives us a factorization of H_{m+1} satisfying the limitation (L_1).

The next step is to check whether $4-$factors can be constructed using $2-$factors obtained from $GH(3, 3^m)$ when m is even.

Theorem 2

Let $3^m \equiv 1 mod(4)$. Then the complete graph K_{3^m} has $2-$factors and $4-$factors.

Proof.

Let $G = K_{3^m}$, where $3^m \equiv 1 mod(4)$. Then G is a $4k-$regular graph. According to the Peterson Theorem it is $2-$factorable since it is $2k_1-$regular graph for $k_1 = 2k$ and number of $2-$factors in a factorization is $\frac{3^m-1}{2}$ which is divisible by 2. Take any two $2-$factors F_1 and F_2, where $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$.

The union $F_1 \cup F_2 = (V, E_1 \cup E_2)$. This leads to form $4-$factors.

3 Results and Discussion

To illustrate the theorem 1, consider the factorization of K_9 using the normalized generalized Hadamard matrix $GH(3,9)$.

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega \\ 1 & 1 & 1 & \omega & \omega & \omega & \omega^2 & \omega^2 & \omega^2 \\ 1 & \omega & \omega^2 & \omega & \omega^2 & 1 & \omega^2 & 1 & \omega \\ 1 & \omega^2 & \omega & \omega & 1 & \omega^2 & \omega^2 & \omega & 1 \\ 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega & \omega & \omega \\ 1 & \omega & \omega^2 & \omega^2 & 1 & \omega & \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega & \omega^2 & \omega & 1 & \omega & 1 & \omega^2 \end{bmatrix}$$

We want to find $2-$factorization $\{F_1, F_2, F_3, F_4\}$ of the complete graph K_9 fulfilling the limitation L_1. Considering all possible combinations, a feasible $2-$factorization of K_9 is (Fig. 3).

$F_1 = \big\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,7\},\{7,8\},\{8,9\},\{9,1\}\big\}$
$F_2 = \big\{\{1,3\},\{3,5\},\{5,7\},\{7,9\},\{9,4\},\{4,2\},\{2,6\},\{6,8\},\{8,1\}\big\}$
$F_3 = \big\{\{1,5\},\{5,9\},\{9,2\},\{2,8\},\{8,4\},\{4,7\},\{7,3\},\{3,6\},\{6,1\}\big\}$
$F_4 = \big\{\{1,4\},\{4,6\},\{6,9\},\{9,3\},\{3,8\},\{8,5\},\{5,2\},\{2,7\},\{7,1\}\big\}$

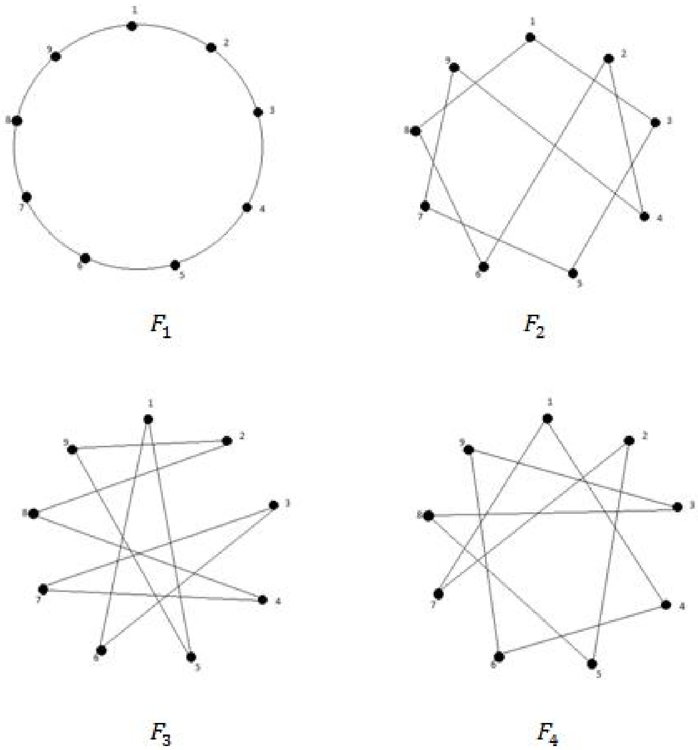

Fig. 3. $2-$ factorization of K_9 obtained from $GH(3, 3^2)$

Note that only $\frac{(n-1)}{2}$ number of rows in the generalized Hadamard matrices have been considered for this construction. That number is the same as the number of $2-$ factors of the complete graph of n vertices. From each row $k = 1,2, \dots, \frac{(n-1)}{2}$, distinct F_k factor can be constructed.

Now consider the construction of $4-$factors of K_9 by using the theorem 2 since $3^2 \equiv 1 mod(4)$. We have already obtained the 4 different $2-$factors. By getting the combinations of $2-$factors using Round Robin tournament schedule and then taking the union of them, 3 different $4-$factors can be constructed. Ultimately, $4-$factorization is obtained.

We can have $\{F_1, F_2\}$ and $\{F_3, F_4\}$ or $\{F_1, F_4\}$ and $\{F_2, F_3\}$ or $\{F_1, F_3\}$ and $\{F_2, F_4\}$ as 3 different $4-$ factors (Fig. 4).

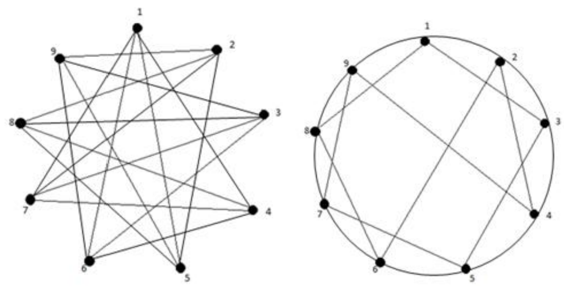

Fig. 4. $\mathbf{4-}$ factorization of K_9 obtained from $GH(3,3^2)$

Let us illustrate the problem of finding a factorization satisfying limitation (L_1) for the following example,

$$H=\begin{bmatrix}1 & 1 & 1 & 1 & 1\\ 1 & \omega & \omega^2 & \omega^3 & \omega^4\\ 1 & \omega^2 & \omega^4 & \omega & \omega^3\\ 1 & \omega^3 & \omega & \omega^4 & \omega^2\\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega\end{bmatrix}$$

We want to find a $1-$factorization $\{F_1,F_2\}$ of the complete graph on 5 vertices K_5 satisfying limitation (L_1). For the rows $(1,\omega,\omega^2,\omega^3,\omega^4)$ and $(1,\omega^2,\omega^4,\omega,\omega^3)$, there are 10 edges we would potentially select $\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\}$, and $\{4,5\}$. However there are 12 possible $2-$factors satisfying limitation (L_1).

$\{\{1,2\},\{1,3\},\{2,4\},\{3,5\},\{4,5\}\}$	$\{\{1,3\},\{1,4\},\{2,3\},\{2,5\},\{4,5\}\}$
$\{\{1,2\},\{1,3\},\{2,5\},\{3,4\},\{4,5\}\}$	$\{\{1,3\},\{1,4\},\{2,5\},\{2,4\},\{3,5\}\}$
$\{\{1,2\},\{1,4\},\{2,3\},\{3,5\},\{4,5\}\}$	$\{\{1,3\},\{1,5\},\{2,4\},\{2,5\},\{3,4\}\}$
$\{\{1,2\},\{1,4\},\{2,5\},\{3,4\},\{3,5\}\}$	$\{\{1,3\},\{1,5\},\{2,3\},\{2,4\},\{4,5\}\}$
$\{\{1,2\},\{1,5\},\{2,4\},\{3,4\},\{3,5\}\}$	$\{\{1,4\},\{1,5\},\{2,3\},\{2,5\},\{3,4\}\}$
$\{\{1,2\},\{1,5\},\{2,3\},\{3,4\},\{4,5\}\}$	$\{\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{3,5\}\}$

Hence, from 12 possible combinations of these pairs of edges a feasible $2-$factorization of K_5 satisfying the requirements is

$F_1=\{\{1,2\},\{1,3\},\{2,4\},\{3,5\},\{4,5\}\}$
$F_2=\{\{1,4\},\{1,5\},\{2,3\},\{2,5\},\{3,4\}\}$

It has already been drawn in Fig.01. Further, we came up with following conjecture.

Conjecture 1: Let p be an odd prime number. Then there exist $2-$ factorization of K_p fulfilling the limitation (L_1).

4 Conclusion

Beyond the work of some of the authors, the present work progresses on the idea of constructing the factorization of complete graphs, focusing on generalized Hadamard matrices. In particular, $2-$factorization of complete graphs $K_n: n=3^m$ is discussed. More specifically, we introduce some limitations. In Particular, $\frac{(n-1)}{2}$ number of rows in the generalized Hadamard matrices are used to form 2-factorization of complete graphs. We discuss some illustrative examples that might be used for studying the factorization of complete graphs. Further, we extend our research to construct $4-$factorization of complete graphs of order 3^m when

m is even. In particular, the following problem might be considered: show whether these constructions can be applied for any generalized Hadamard matrix of order p^m, where p is odd prime and $m > 1$. This will be the concern of our future work by automating the results we obtained.

Competing Interests

Authors have declared that no competing interests exist.

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