Engineering and Applied Science Letters

PSR Press

Article

Reverse Hermite-Hadamard's inequalities using ψ-fractional integral

Tariq A. Aljaaidi[1,*] and Deepak B. Pachpatte[1]

[1] Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, (M.S), 431001, India.
* Correspondence: tariq10011@gmail.com

Received: 4 July 2020; Accepted: 30 November 2020; Published: 21 December 2020.

Abstract: Our purpose in this paper is to use $\psi-$Riemann-Liouville fractional integral operator which is the fractional integral of any function with respect to another increasing function to establish some new fractional integral inequalities of Hermite-Hadamard, involving concave functions. Using the concave functions, we establish some new fractional integral inequalities related to the Hermite-Hadamard type inequalities via $\psi-$Riemann-Liouville fractional integral operator.

Keywords: Fractional inequalities, ψ-Riemann-Liouville fractional integral, ψ-Riemann-Liouville derivative.

1. Introduction

The classical calculus of derivatives and integrals which involves integer orders is extended with fractional orders that belong to the real numbers. In last few decades, the fractional calculus theory receives more attention due to its significant applications in several scopes such as physics, fluid dynamics, computer networking, image processing, biology, signal processing, control theory and other scopes. Because of the importance of fractional calculus, many researchers have shown their intense interest. One of the prevalent approaches among researchers is the use of fractional derivatives and integral operators. As a consequence, several distinct kinds of fractional integrals and derivatives operators have been realized, such as the Liouville, Riemann-Liouville, Katugampola, Weyl types, Hadamard and some other types which can be found in Kilbas *et al.*, [1].

Hilfer [2] in (2000), through his contribution to improve the fractional calculus, established a new fractional derivative operator for any real order δ, which gives the Caputo derivative and the Riemann-Liouville fractional operator. The primary concept and properties and more information of ψ-Riemann–Liouville fractional derivative and integral can be found in [1]. In (2017), Almeida [3], introduced ψ-Caputo fractional derivative and investigated its significant properties. Recently, in 2018, Sousa and Oliveira [4], introduced a generalization of many existing fractional derivative operators called ψ-Hilfer derivative.

The mathematical inequalities play a very reliable role in classical integral and differential equations as well as in the past few years, many of useful mathematical inequalities have been originated by many authors, see [5–8]. One of the most significant integral inequalities is that discovered by Hermite [9] and Hadamard [10] for convex function f as follows

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{a-b}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

If f is a concave function then both inequalities in (1) are held in a reversed direction. For some historical of Hermite-Hadamard inequalities [11] and the references therein. In the last few decades, these inequalities have been received a considerable attention by many authors and several articles have appeared in the literature, see [12–14]. In 2010, Dahmani [15], studied the Hermite-Hadamard type inequalities for concave functions by means of Riemann–Liouville fractional integral. Sarikaya *et al.*, in 2013 [16], gave the Hermite-Hadamard type inequalities for convex function using Riemann–Liouville fractional integral. In 2014, Set *et al.*, established Hermite-Hadamard type inequalities for s-convex functions in the second sense proved by Dragomir *et al.*, [17] and m-convex functions via fractional integrals. In (2015), Noor *et al.*, [18], derived some quantum estimates for Hermite–Hadamard inequalities for q-differentiable quasi convex functions and q-differentiable convex functions. Liu *et al.*, [19] in 2016, introduced some inequalities of Hermite-Hadamard type for MT-convex functions using classical integrals and Riemann-Liouville fractional integrals. In 2017, Agarwal *et al.*, [20], obtained some Hermite-Hadamard type inequalities for convex functions via $(k,s)-$Riemann-Liouville fractional integrals. Muhammad A. Khan [21] in 2018, proved new Hermite-Hadamard inequalities for convex function, $s-$convex and coordinate convex functions by using conformable fractional integrals. Recently in 2019, a lots of researchers studied Hermite-Hadamard inequalities for several kinds of convexity of the functions, for more details we refer the readers to see [22,23]. Very recently, in 2020, Chudziak and Ołdak introduced notion of a co-ordinated (F,G)-convex function defined on an interval in $\mathbb{R}^2$.

The main objective of this paper is to establish some new fractional integral Hermite-Hadamard inequalities for concave functions by using $\psi-$Riemann-Liouville fractional integral operator. Moreover, we introduce some new fractional integral inequalities related to the Hermite-Hadamard inequalities via $\psi-$Riemann-Liouville fractional integral operator. The paper is organized as follows: In Section 2, we collect some notations, definitions, results and preliminary facts which are used throughout this paper. In Section 3, we present the reverse Hermite-Hadamard's inequalities for concave functions. In Section 4, we give some other related results of Hermite-Hadamard type inequalities which involving $\psi-$Riemann-Liouville fractional integral operator.

2. Basic definitions and tools

This section is dedicated for some basic definitions and properties of fractional integrals used to obtain and discuss our new results. We also outline some basic results related to this work.

Let $\delta > 0$, $m \in \mathbb{N}$, with $\Upsilon = [a,b]$ $(-\infty \leq a < t < b \leq \infty)$, be a finite or infinite interval. Assume that f be an integrable function defined on Υ and $\psi : \Upsilon \to \mathbb{R}$ be an increasing function for all $t \in \Upsilon$, which belong to $C^1(\Upsilon, \mathbb{R})$ with condition that $\psi'(t)$ must be nonzero along the interval Υ. The $\psi-$Riemann-Liouville fractional derivative of order δ of a function f are defined by [1,24]:

$$\begin{aligned} D_{a^+}^{\delta;\psi} f(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^m \mathcal{I}_{a^+}^{m-\delta;\psi} f(t) \\ &= \frac{1}{\Gamma(m-\delta)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^m \int_a^t \psi'(\xi)\left[\psi(t)-\psi(\xi)\right]^{m-\delta-1} f(\xi)\, d\xi, \end{aligned} \tag{2}$$

and

$$\begin{aligned} D_{b^-}^{\delta;\psi} f(t) &= \left(-\frac{1}{\psi'(t)}\frac{d}{dt}\right)^m \mathcal{I}_{b^-}^{m-\delta;\psi} f(t) \\ &= \frac{1}{\Gamma(m-\delta)}\left(-\frac{1}{\psi'(t)}\frac{d}{dt}\right)^m \int_t^b \psi'(\xi)\left[\psi(\xi)-\psi(t)\right]^{m-\delta-1} f(\xi)\, d\xi. \end{aligned} \tag{3}$$

Definition 1. Let $\delta > 0$ and f be an integrable function defined on Υ and $\psi(t) \in C^1(\Upsilon, \mathbb{R})$ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in \Upsilon$. The left and right $\psi-$Riemann-Liouville fractional integral of order δ with respect to the function ψ of a function f are respectively defined by [1,24]:

$$\mathcal{I}_{a^+}^{\delta;\psi} f(t) = \frac{1}{\Gamma(\delta)} \int_a^t \psi'(\xi)\left[\psi(t)-\psi(\xi)\right]^{\delta-1} f(\xi)\, d\xi, \tag{4}$$

and

$$\mathcal{I}_{b^-}^{\delta;\psi} f(t) = \frac{1}{\Gamma(\delta)} \int_t^b \psi'(\xi)\left[\psi(\xi)-\psi(t)\right]^{\delta-1} f(\xi)\, d\xi. \tag{5}$$

Lemma 1. *[1] Let $\delta > 0$ and $\mu > 0$. If $f(t) = [\psi(t) - \psi(\xi)]^{\mu-1}$, then*

$$\mathcal{I}_{a^+}^{\delta;\psi} f(t) = \frac{\Gamma(\mu)}{\Gamma(\delta+\mu)}\left[\psi(t)-\psi(\xi)\right]^{\delta+\mu-1}.$$

Definition 2. The function $f : (\Lambda \subseteq \mathbb{R}) \longrightarrow \mathbb{R}$ is said to be concave function if the following inequality holds

$$f(\lambda x + (1-\lambda) y) \geq \lambda f(x) + (1-\lambda) f(y), \tag{6}$$

for all $x, y \in \Lambda$ and $\lambda \in [0,1]$. We say that f is convex if the inequality (6) is reversed.

Theorem 3. *[16] Let $f : [a,b] \longrightarrow \mathbb{R}$ be a convex positive function on $[a,b]$ with $0 \leq a < b$, then for all $\delta > 0$, the following inequality holds:*

$$\frac{f(a)+f(b)}{2} \leq \frac{\Gamma(\delta+1)}{2(a+b)^{\delta}} \left[\mathcal{I}^{\delta}_{a+} f(b) + \mathcal{I}^{\delta}_{b-} f(a)\right] \leq f\left(\frac{a+b}{2}\right). \tag{7}$$

Lemma 2. *[15] Let $h : [a,b] \longrightarrow \mathbb{R}$ be a concave. Then the following inequality holds:*

$$h(a) + h(b) \leq h[a+b-t] + h(t) \leq 2h\left(\frac{a+b}{2}\right).$$

Theorem 4. *[15] Let f and g be two positive functions on $[0,\infty)$. If f and g are a concave functions on $[0,\infty)$, then for all $p > 1$, $q > 1$ and $\delta > 0$, the following inequality holds:*

$$2^{-p-q}[f(0)+f(x)]^{p}[g(0)+g(x)]^{q}\left(\mathcal{I}^{\delta} x^{\delta-1}\right)^{2} \leq \mathcal{I}^{\delta}\left[x^{\delta-1} f^{p}(x)\right] \mathcal{I}^{\delta}\left(x^{\delta-1} g^{q}(x)\right).$$

Theorem 5. *[15] Let f and g be two positive functions on $[0,\infty)$. If f and g are a concave functions on $[0,\infty)$, then for all $p > 1$, $q > 1$ and $\delta > 0$, $\sigma > 0$ the following inequality holds:*

$$\begin{aligned}&2^{2-p-q}[f(0)+f(x)]^{p}[g(0)+g(x)]^{q}\left(\mathcal{I}\left({}^{\delta} x^{\sigma-1}\right)\right)^{2}\\ &\leq \frac{\Gamma(\sigma)}{\Gamma(\delta)}\mathcal{I}^{\sigma}\left[x^{\sigma-1} f^{p}(x)\right] + \mathcal{I}^{\delta}\left[x^{\sigma-1} f^{p}(x)\right] \frac{\Gamma(\sigma)}{\Gamma(\delta)}\mathcal{I}^{\sigma}\left[x^{\sigma-1} g^{q}(x)\right] + \mathcal{I}^{\delta}\left[x^{\sigma-1} g^{q}(x)\right].\end{aligned}$$

3. The reverse Hermite-Hadamard's inequalities for fractional integral

Now, we give the reverse Hermite-Hadamard's inequalities involving concave functions for ψ-Riemann-Liouville fractional integral operators.

Theorem 6. *Let $\psi : [a,b] \longrightarrow \Lambda \subseteq \mathbb{R}$ with $0 \leq a < b$ be an increasing and bijective function having a continuous derivative $\psi'(x) \neq 0 \ \forall\, x \in [a,b]$, $\psi(0) = 0$, $\psi(1) = 1$ and $f : \Lambda \longrightarrow \mathbb{R}$ be an increasing and differentiable function on Λ° such that $(f \circ \psi) : [a,b] \longrightarrow \mathbb{R}$ be an integrable mapping on $[a,b]$. If ψ and f are concave and positive functions, then the following inequality holds:*

$$f\left(\frac{\psi(a)+\psi(b)}{2}\right) \geq \frac{\Gamma(\delta+1)}{2[\psi(a)+\psi(b)]^{\delta}}\left[\mathcal{I}^{\delta;\psi}_{a^+}(f\circ\psi)(b) + \mathcal{I}^{\delta;\psi}_{b^-}(f\circ\psi)(a)\right] \geq \frac{(f\circ\psi)(a)+(f\circ\psi)(b)}{2}. \tag{8}$$

Proof. For any $x, y \in [a,b]$, using the concavity of f and ψ, we have

$$\begin{aligned}(f\circ\psi)[\lambda x + (1-\lambda) y] &= f[\psi[\lambda x + (1-\lambda) y]]\\ &\geq f[\lambda\psi(x) + (1-\lambda)\psi(y)]\\ &\geq \lambda f[\psi(x)] + (1-\lambda) f[\psi(y)]\\ &= \lambda(f\circ\psi)(x) + (1-\lambda)(f\circ\psi)(y).\end{aligned} \tag{9}$$

Putting $\lambda = \frac{1}{2}$ and using (9), we can write

$$f\left(\frac{\psi(x)+\psi(y)}{2}\right) \geq \frac{(f\circ\psi)(x)+(f\circ\psi)(y)}{2}. \tag{10}$$

Let

$$\psi(x)=\psi(t)\,\psi(a)+[1-\psi(t)]\,\psi(b)\,, \tag{11}$$

and

$$\psi(y)=[1-\psi(t)]\,\psi(a)+\psi(t)\,\psi(b)\,, \tag{12}$$

where x,y are variables containing t. By substituting (11) and (12) in (10), we get

$$2f\left(\frac{\psi(a)+\psi(b)}{2}\right)\geq f\left[\psi(t)\,\psi(a)+[1-\psi(t)]\,\psi(b)\right]+f\left[[1-\psi(t)]\,\psi(a)+\psi(t)\,\psi(b)\right]. \tag{13}$$

Now, multiplying both sides of (13) by $\psi'(t)\,\psi^{\delta-1}(t)$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}\frac{2}{\delta}f\left(\frac{\psi(a)+\psi(b)}{2}\right) &\geq \int_0^1\psi'(t)\,\psi^{\delta-1}(t)\,f\left[\psi(t)\,\psi(a)+[1-\psi(t)]\,\psi(b)\right]dt\\ &\quad+\int_0^1\psi'(t)\,\psi^{\delta-1}(t)\,f\left[[1-\psi(t)]\,\psi(a)+\psi(t)\,\psi(b)\right]dt.\\ &= I_1+I_2.\end{aligned} \tag{14}$$

From (11), we have

$$\frac{d}{dt}\psi(x)=\frac{d}{dt}\left[\psi(t)\,\psi(a)+[1-\psi(t)]\,\psi(b)\right]\Longrightarrow\frac{\psi'(x)\,dx}{\psi(a)-\psi(b)}=\psi'(t)\,dt,$$

and

$$\psi(t)=\frac{\psi(x)-\psi(b)}{\psi(a)-\psi(b)}.$$

So, we have

$$\begin{aligned}I_1 &= \int_b^a\left(\frac{\psi(x)-\psi(b)}{\psi(a)-\psi(b)}\right)^{\delta-1}f(\psi(x))\,\frac{\psi'(x)\,dx}{\psi(a)-\psi(b)}\\ &= \int_a^b\psi'(x)\left(\frac{\psi(b)-\psi(x)}{\psi(b)-\psi(a)}\right)^{\delta-1}(f\circ\psi)(x)\,\frac{dx}{\psi(b)-\psi(a)}\\ &= \frac{\Gamma(\delta)}{[\psi(b)-\psi(a)]^{\delta}}\mathcal{I}_{a^+}^{\delta;\psi}(f\circ\psi)(b)\,.\end{aligned} \tag{15}$$

Also from (12), we have

$$\frac{d}{dt}\psi(y)=\frac{d}{dt}\left[[1-\psi(t)]\,\psi(a)+\psi(t)\,\psi(b)\right]\Longrightarrow\frac{\psi'(y)\,dy}{[\psi(b)-\psi(a)]}=\psi'(t)\,dt,$$

and

$$\psi(t)=\frac{\psi(y)-\psi(a)}{\psi(b)-\psi(a)}.$$

So, we have

$$\begin{aligned}I_2 &= \int_a^b\left(\frac{\psi(y)-\psi(a)}{\psi(b)-\psi(a)}\right)^{\delta-1}f(\psi(y))\,\frac{\psi'(y)\,dy}{[\psi(b)-\psi(a)]}\\ &= \frac{1}{[\psi(b)-\psi(a)]^{\delta}}\int_a^b\psi'(y)\,[\psi(y)-\psi(a)]^{\delta-1}(f\circ\psi)(y)\,dy\\ &= \frac{\Gamma(\delta)}{[\psi(b)-\psi(a)]^{\delta}}\mathcal{I}_{b^-}^{\delta;\psi}(f\circ\psi)(a)\,.\end{aligned} \tag{16}$$

Using (14), (15) and (16), we get

$$f\left(\frac{\psi(a)+\psi(b)}{2}\right) \geq \frac{\Gamma(\delta+1)}{2[\psi(b)-\psi(a)]^{\delta}}\left[\mathcal{I}_{a^+}^{\delta;\psi}(f\circ\psi)(b)+\mathcal{I}_{b^-}^{\delta;\psi}(f\circ\psi)(a)\right], \tag{17}$$

which is the first inequality in (8). To prove the second inequality and using the concavity of f and ψ, we can write for $\lambda \in [0,1]$

$$f[\psi(t)\psi(a)+[1-\psi(t)]\psi(b)] \geq \psi(t)(f\circ\psi)(a)+[1-\psi(t)](f\circ\psi)(b), \tag{18}$$

and

$$f[[1-\psi(t)]\psi(a)+\psi(t)\psi(b)] \geq [1-\psi(t)](f\circ\psi)(a)+\psi(t)(f\circ\psi)(b). \tag{19}$$

Adding (18) and (19), we obtain

$$f[\psi(t)\psi(a)+[1-\psi(t)]\psi(b)]+f[[1-\psi(t)]\psi(a)+\psi(t)\psi(b)] \geq (f\circ\psi)(b)+(f\circ\psi)(a). \tag{20}$$

Now, multiplying both sides of (20) by $\psi'(t)\psi^{\delta-1}(t)$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\frac{\Gamma(\delta)}{[\psi(b)-\psi(a)]^{\delta}}\left[\mathcal{I}_{a^+}^{\delta;\psi}(f\circ\psi)(b)+\mathcal{I}_{b^-}^{\delta;\psi}(f\circ\psi)(a)\right] \geq \frac{(f\circ\psi)(b)+(f\circ\psi)(a)}{\delta}. \tag{21}$$

Hence, by combining the inequalities (17) and (20), we get the desired inequality (8). □

Remark 1. (i) If we put $\psi(x)=x$, $\forall x\in[a,b]$, for f a convex function, then both inequalities (8) reversed and Theorem 6 reduce to Theorem 3 obtained by Sarikaya *et al.*, in [16] for classical Riemann-Liouville fractional integral.

(ii) Applying inequalities (8) for $\psi(x)=x$, $\forall x\in[a,b]$ and $\delta=1$, for f be a convex function, we obtain the classical Hermite-Hadamard inequalities (1).

4. Hermite-Hadamard type inequalities for fractional integral

In this section, we generalize some Hermite-Hadamard type inequalities involving concave functions introduced by Dahmani [15] using the Riemann-Liouville fractional integral with respect to other monotone and bijective function. In present part, we use only the left-sided fractional integrals (4). Moreover, we consider $a=0$ to obtain and discuss our results. We first prove the following lemma:

Lemma 3. *Let $\psi:[0,\infty)\longrightarrow\Lambda$ be an increasing and bijective function having a continuous derivative $\psi'(t)\neq 0$ $\forall\, t\in[0,\infty)$, $\psi(0)=0$, $\psi(1)=1$ and $h:\Lambda\longrightarrow\mathbb{R}$ be an increasing and differentiable function on Λ° such that $(h\circ\psi):[0,\infty)\longrightarrow\mathbb{R}$ be an integrable mapping on $[0,\infty)$. If h is a concave functions on Λ, then we have*

$$(h\circ\psi)(c)+(h\circ\psi)(d) \leq h[\psi(c)+\psi(d)-\psi(t)]+(h\circ\psi)(t) \leq 2h\left(\frac{\psi(c)+\psi(d)}{2}\right). \tag{22}$$

Proof. Since h be a concave function on Λ, so for any $c,d\in[0,\infty)$, we can write

$$h\left(\frac{\psi(c)+\psi(d)}{2}\right)=h\left(\frac{\psi(c)+\psi(d)+\psi(t)-\psi(t)}{2}\right) \geq \frac{h[\psi(c)+\psi(d)-\psi(t)]+h[\psi(t)]}{2}. \tag{23}$$

If we choose $\psi(t)=\lambda\psi(c)+[1-\lambda]\psi(d)$, then we have

$$\begin{aligned}&\frac{1}{2}[h[\psi(c)+\psi(d)-\lambda\psi(c)-[1-\lambda]\psi(d)]+h[\lambda\psi(c)+[1-\lambda]\psi(d)]]\\ &=\frac{1}{2}[h[\lambda\psi(d)+[1-\lambda]\psi(c)]+h[\lambda\psi(c)+[1-\lambda]\psi(d)]].\end{aligned}$$

Using the concavity of h, we obtain

$$\frac{1}{2}\left[h\left[\lambda\psi\left(d\right)+\left[1-\lambda\right]\psi\left(c\right)\right]+h\left[\lambda\psi\left(c\right)+\left[1-\lambda\right]\psi\left(d\right)\right]\right]\geq\frac{1}{2}\left[\left(h\circ\psi\right)\left(c\right)+\left(h\circ\psi\right)\left(d\right)\right]. \tag{24}$$

By (23) and (24), we get

$$\left(h\circ\psi\right)\left(c\right)+\left(h\circ\psi\right)\left(d\right)\leq h\left[\psi\left(c\right)+\psi\left(d\right)-\psi\left(t\right)\right]+\left(h\circ\psi\right)\left(t\right)\leq 2h\left(\frac{\psi\left(c\right)+\psi\left(d\right)}{2}\right),$$

which is the required inequality (22). □

Theorem 7. *Let $\psi:[0,\infty)\longrightarrow\Lambda$ be an increasing positive and bijective function having a continuous derivative $\psi'\left(x\right)\neq 0\ \forall\, x\in[0,\infty)$, $\psi\left(0\right)=0$, $\psi\left(1\right)=1$ and $f,g:\Lambda\longrightarrow\mathbb{R}$ be an increasing and differentiable functions on Λ° such that $\left(f\circ\psi\right),\left(g\circ\psi\right):[0,\infty)\longrightarrow\mathbb{R}$ are two integrable mappings on $[0,\infty)$. If f and g are a concave functions on Λ. Then for all $p>1$, $q>1$ and $\delta>0$, the following inequality holds:*

$$\begin{aligned}&2^{-p-q}\left[f\left(0\right)+\left(f\circ\psi\right)\left(x\right)\right]^{p}\left[g\left(0\right)+\left(g\circ\psi\right)\left(x\right)\right]^{q}\left(\mathcal{I}^{\delta;\psi}\psi^{\delta-1}\left(x\right)\right)^{2}\\&\leq\mathcal{I}^{\delta;\psi}\left[\psi^{\delta-1}\left(x\right)\left(f\circ\psi\right)^{p}\left(x\right)\right]\mathcal{I}^{\delta;\psi}\left[\psi^{\delta-1}\left(x\right)\left(g\circ\psi\right)^{q}\left(x\right)\right].\end{aligned} \tag{25}$$

Proof. Since f^{p} and g^{q} are a concave functions on Λ, so by Lemma (3), for any $x,y>0$, we have

$$f^{p}\left(0\right)+\left(f\circ\psi\right)^{p}\left(x\right)\leq f^{p}\left[\psi\left(x\right)-\psi\left(y\right)\right]+\left(f\circ\psi\right)^{p}\left(y\right)\leq 2\left(f\circ\psi\right)^{p}\left(\frac{x}{2}\right), \tag{26}$$

and

$$g^{q}\left(0\right)+\left(g\circ\psi\right)^{q}\left(x\right)\leq g^{q}\left[\psi\left(x\right)-\psi\left(y\right)\right]+\left(g\circ\psi\right)^{q}\left(y\right)\leq 2\left(g\circ\psi\right)^{q}\left(\frac{x}{2}\right). \tag{27}$$

Multiplying both sides of (26) and (27) by $\frac{\psi'(y)}{\Gamma(\delta)}\left[\psi\left(x\right)-\psi\left(y\right)\right]^{\delta-1}\psi^{\delta-1}\left(y\right)$, $y\in(0,x)$ and integrating the resulting inequalities with respect to y over $(0,x)$, we obtain

$$\begin{aligned}&\frac{f^{p}\left(0\right)+\left(f\circ\psi\right)^{p}\left(x\right)}{\Gamma\left(\delta\right)}\int_{0}^{x}\psi'\left(y\right)\left[\psi\left(x\right)-\psi\left(y\right)\right]^{\delta-1}\psi^{\delta-1}\left(y\right)dy\\&\leq\frac{1}{\Gamma\left(\delta\right)}\int_{0}^{x}\psi'\left(y\right)\left[\psi\left(x\right)-\psi\left(y\right)\right]^{\delta-1}\psi^{\delta-1}\left(y\right)f^{p}\left[\psi\left(x\right)-\psi\left(y\right)\right]dy\\&\quad+\frac{1}{\Gamma\left(\delta\right)}\int_{0}^{x}\psi'\left(y\right)\left[\psi\left(x\right)-\psi\left(y\right)\right]^{\delta-1}\psi^{\delta-1}\left(y\right)\left(f\circ\psi\right)^{p}\left(y\right)dy\\&\leq\frac{2\left(f\circ\psi\right)^{p}\left(\frac{x}{2}\right)}{\Gamma\left(\delta\right)}\int_{0}^{x}\psi'\left(y\right)\left[\psi\left(x\right)-\psi\left(y\right)\right]^{\delta-1}\psi^{\delta-1}\left(y\right)dy,\end{aligned} \tag{28}$$

and

$$\begin{aligned}&\frac{g^{q}\left(0\right)+\left(g\circ\psi\right)^{q}\left(x\right)}{\Gamma\left(\delta\right)}\int_{0}^{x}\psi'\left(y\right)\left[\psi\left(x\right)-\psi\left(y\right)\right]^{\delta-1}\psi^{\delta-1}\left(y\right)dy\\&\leq\frac{1}{\Gamma\left(\delta\right)}\int_{0}^{x}\psi'\left(y\right)\left[\psi\left(x\right)-\psi\left(y\right)\right]^{\delta-1}\psi^{\delta-1}\left(y\right)g^{q}\left[\psi\left(x\right)-\psi\left(y\right)\right]dy\\&\quad+\frac{1}{\Gamma\left(\delta\right)}\int_{0}^{x}\psi'\left(y\right)\left[\psi\left(x\right)-\psi\left(y\right)\right]^{\delta-1}\psi^{\delta-1}\left(y\right)\left(g\circ\psi\right)^{q}\left(y\right)dy\\&\leq\frac{2\left(g\circ\psi\right)^{q}\left(\frac{x}{2}\right)}{\Gamma\left(\delta\right)}\int_{0}^{x}\psi'\left(y\right)\left[\psi\left(x\right)-\psi\left(y\right)\right]^{\delta-1}\psi^{\delta-1}\left(y\right)dy.\end{aligned} \tag{29}$$

Using the change of variable $\psi\left(u\right)=\psi\left(x\right)-\psi\left(y\right)$, where $u\in[0,\infty)$ is a variable containing y, we have

$$\frac{d}{dy}\left[\psi\left(u\right)\right]=\frac{d}{dy}\left[\psi\left(x\right)-\psi\left(y\right)\right]\Longrightarrow\psi'\left(u\right)du=-\psi'\left(y\right)dy.$$

Then, we can write

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_0^x \psi'(y)\left[\psi(x)-\psi(y)\right]^{\delta-1}\psi^{\delta-1}(y)\, f^p\left[\psi(x)-\psi(y)\right]dy\\&=\frac{1}{\Gamma(\delta)}\int_0^x \psi'(u)\left[\psi(x)-\psi(u)\right]^{\delta-1}\psi^{\delta-1}(u)\,(f\circ\psi)^p(u)\,du=\mathcal{I}^{\delta;\psi}\left[\psi^{\delta-1}(x)\,(f\circ\psi)^p(x)\right],\end{aligned}\tag{30}$$

and

$$\begin{aligned}&\frac{1}{\Gamma(\delta)}\int_0^x \psi'(y)\left[\psi(x)-\psi(y)\right]^{\delta-1}\psi^{\delta-1}(y)\, g^q\left[\psi(x)-\psi(y)\right]dy\\&=\frac{1}{\Gamma(\delta)}\int_0^x \psi'(u)\left[\psi(x)-\psi(u)\right]^{\delta-1}\psi^{\delta-1}(u)\,(g\circ\psi)^q(u)\,du=\mathcal{I}^{\delta;\psi}\left[\psi^{\delta-1}(x)\,(g\circ\psi)^q(x)\right].\end{aligned}\tag{31}$$

Now, by using (28) and (30), we get

$$\left[f^p(0)+(f\circ\psi)^p(x)\right]\mathcal{I}^{\delta;\psi}\psi^{\delta-1}(x)\le 2\mathcal{I}^{\delta;\psi}\left[\psi^{\delta-1}(x)\,(f\circ\psi)^p(x)\right]\le 2\,(f\circ\psi)^p\left(\frac{x}{2}\right)\mathcal{I}^{\delta;\psi}\psi^{\delta-1}(x),\tag{32}$$

and using (29) and (31), we get

$$\left[g^q(0)+(g\circ\psi)^q(x)\right]\mathcal{I}^{\delta;\psi}\psi^{\delta-1}(x)\le 2\mathcal{I}^{\delta;\psi}\left[\psi^{\delta-1}(x)\,(g\circ\psi)^q(x)\right]\le 2\,(g\circ\psi)^q\left(\frac{x}{2}\right)\mathcal{I}^{\delta;\psi}\psi^{\delta-1}(x).\tag{33}$$

The inequalities (32) and (33) yields

$$\begin{aligned}&\left[f^p(0)+(f\circ\psi)^p(x)\right]\left[g^q(0)+(g\circ\psi)^q(x)\right]\left(\mathcal{I}^{\delta;\psi}\psi^{\delta-1}(x)\right)^2\\&\le 4\mathcal{I}^{\delta;\psi}\left[\psi^{\delta-1}(x)\,(f\circ\psi)^p(x)\right]\mathcal{I}^{\delta;\psi}\left[\psi^{\delta-1}(x)\,(g\circ\psi)^q(x)\right].\end{aligned}\tag{34}$$

On the other hand, we have f and g are positive functions and ψ is increasing function on $[0,\infty)$. Then for any $x>0$, $p\ge 1$, $q\ge 1$, we can write

$$\left[\frac{f^p(0)+(f\circ\psi)^p(x)}{2}\right]^{\frac{1}{p}}\ge\frac{f(0)+(f\circ\psi)(x)}{2},\tag{35}$$

and

$$\left[\frac{g^q(0)+(g\circ\psi)^q(x)}{2}\right]^{\frac{1}{q}}\ge\frac{g(0)+(g\circ\psi)(x)}{2}.\tag{36}$$

Multiplying both sides of (35) and (36) by $\frac{\psi'(y)}{\Gamma(\delta)}\left[\psi(x)-\psi(y)\right]^{\delta-1}\psi^{\delta-1}(y)\,, y\in(0,x)$, then integrating the resulting inequalities with respect to y over $(0,x)$, we get

$$\frac{f^p(0)+(f\circ\psi)^p(x)}{2}\mathcal{I}^{\delta;\psi}\psi^{\delta-1}(x)\ge 2^{-p}\left[f(0)+(f\circ\psi)(x)\right]^p\mathcal{I}^{\delta;\psi}\psi^{\delta-1}(x),\tag{37}$$

and

$$\frac{g^q(0)+(g\circ\psi)^q(x)}{2}\mathcal{I}^{\delta;\psi}\psi^{\delta-1}(x)\ge 2^{-q}\left[g(0)+(g\circ\psi)(x)\right]^q\mathcal{I}^{\delta;\psi}\psi^{\delta-1}(x).\tag{38}$$

The inequalities (37) and (38) yields

$$\begin{aligned}&\frac{1}{4}\left[f^p(0)+(f\circ\psi)^p(x)\right]\left[g^q(0)+(g\circ\psi)^q(x)\right]\left(\mathcal{I}^{\delta;\psi}\psi^{\delta-1}(x)\right)^2\\&\ge 2^{-p-q}\left[f(0)+(f\circ\psi)(x)\right]^p\left[g(0)+(g\circ\psi)(x)\right]^q\left(\mathcal{I}^{\delta;\psi}\psi^{\delta-1}(x)\right)^2.\end{aligned}\tag{39}$$

Combining the inequalities (34) and (39), we obtain the desired inequality (25). □

Remark 2. (i) If we put $\psi(x)=x$ for all $x\in[0,\infty)$, then Lemma 3 reduce to Lemma 2 and Theorem 7 reduce to Theorem 4 obtained by Dahmani in [15].

(ii) Applying Theorem 7 for $\psi(x) = x$ for all $x \in [0,\infty)$, $\delta = 1$, we obtain Theorem 5 obtained by Set *et al.*, in [25].

No, we give the following version of Theorem 7 with two parameters for ψ-Riemann-Liouville fractional integral operator.

Theorem 8. *Let $\psi : [0,\infty) \longrightarrow \Lambda$ be an increasing positive and bijective function having a continuous derivative $\psi'(x) \neq 0$ $\forall x \in [0,\infty)$, $\psi(0) = 0$, $\psi(1) = 1$ and $f, g : \Lambda \longrightarrow \mathbb{R}$ be an increasing and differentiable functions on Λ° such that $(f \circ \psi), (g \circ \psi) : [0,\infty) \longrightarrow \mathbb{R}$ are two integrable mappings on $[0,\infty)$. If f and g are a concave functions on Λ. Then for all $p > 1$, $q > 1$ and $\delta > 0$, $\sigma > 0$, the following inequality holds:*

$$\begin{aligned} &2^{2-p-q}\left[f(0) + (f\circ\psi)(x)\right]^p \left[g(0) + (g\circ\psi)(x)\right]^q \left(\mathcal{I}^{\delta;\psi}\psi^{\sigma-1}(x)\right)^2 \\ &\leq \frac{\Gamma(\sigma)}{\Gamma(\delta)}\mathcal{I}^{\sigma;\psi}\left[\psi^{\delta-1}(x)(f\circ\psi)^p(x)\right] + \mathcal{I}^{\delta;\psi}\left[\psi^{\sigma-1}(x)(f\circ\psi)^p(x)\right] \\ &\quad\times \frac{\Gamma(\sigma)}{\Gamma(\delta)}\mathcal{I}^{\sigma;\psi}\left[\psi^{\delta-1}(x)(g\circ\psi)^q(x)\right] + \mathcal{I}^{\delta;\psi}\left[\psi^{\sigma-1}(x)(g\circ\psi)^q(x)\right]. \end{aligned} \tag{40}$$

Proof. By using Lemma 3 and as f^p and g^q are concave functions on Λ, then we have for any $x, y > 0$

$$f^p(0) + (f\circ\psi)^p(x) \leq f^p[\psi(x) - \psi(y)] + (f\circ\psi)^p(y) \leq 2(f\circ\psi)^p\left(\frac{x}{2}\right), \tag{41}$$

and

$$g^q(0) + (g\circ\psi)^q(x) \leq g^q[\psi(x) - \psi(y)] + (g\circ\psi)^q(y) \leq 2(g\circ\psi)^q\left(\frac{x}{2}\right). \tag{42}$$

Now, multiplying both sides of (41) and (42) by $\frac{\psi'(y)}{\Gamma(\delta)}[\psi(x) - \psi(y)]^{\delta-1}\psi^{\sigma-1}(y)$, $y \in (0,x)$, then integrating the resulting inequalities with respect to y over $(0,x)$, we obtain

$$\begin{aligned} &\frac{f^p(0) + (f\circ\psi)^p(x)}{\Gamma(\delta)}\int_0^x \psi'(y)[\psi(x) - \psi(y)]^{\delta-1}\psi^{\sigma-1}(y)\,dy \\ &\leq \frac{1}{\Gamma(\delta)}\int_0^x \psi'(y)[\psi(x) - \psi(y)]^{\delta-1}\psi^{\sigma-1}(y) f^p[\psi(x) - \psi(y)]\,dy \\ &\quad+ \frac{1}{\Gamma(\delta)}\int_0^x \psi'(y)[\psi(x) - \psi(y)]^{\delta-1}\psi^{\sigma-1}(y)(f\circ\psi)^p(y)\,dy \\ &\leq \frac{2(f\circ\psi)^p\left(\frac{x}{2}\right)}{\Gamma(\delta)}\int_0^x \psi'(y)[\psi(x) - \psi(y)]^{\delta-1}\psi^{\sigma-1}(y)\,dy, \end{aligned} \tag{43}$$

and

$$\begin{aligned} &\frac{g^q(0) + (g\circ\psi)^q(x)}{\Gamma(\delta)}\int_0^x \psi'(y)[\psi(x) - \psi(y)]^{\delta-1}\psi^{\sigma-1}(y)\,dy \\ &\leq \frac{1}{\Gamma(\delta)}\int_0^x \psi'(y)[\psi(x) - \psi(y)]^{\delta-1}\psi^{\sigma-1}(y) g^q[\psi(x) - \psi(y)]\,dy \\ &\quad+ \frac{1}{\Gamma(\delta)}\int_0^x \psi'(y)[\psi(x) - \psi(y)]^{\delta-1}\psi^{\sigma-1}(y)(g\circ\psi)^q(y)\,dy \\ &\leq \frac{2(g\circ\psi)^q\left(\frac{x}{2}\right)}{\Gamma(\delta)}\int_0^x \psi'(y)[\psi(x) - \psi(y)]^{\delta-1}\psi^{\sigma-1}(y)\,dy\,. \end{aligned} \tag{44}$$

Using the change of variable $\psi(u) = \psi(x) - \psi(y)$, where $u \in [0,\infty)$ is a variable containing y, we have

$$\begin{aligned} &\frac{1}{\Gamma(\delta)}\int_0^x \psi'(y)[\psi(x) - \psi(y)]^{\delta-1}\psi^{\delta-1}(y) f^p[\psi(x) - \psi(y)]\,dy \\ &= \frac{\Gamma(\sigma)}{\Gamma(\sigma)\Gamma(\delta)}\int_0^x \psi'(u)[\psi(x) - \psi(u)]^{\sigma-1}\psi^{\delta-1}(u)(f\circ\psi)^p(u)\,du = \frac{\Gamma(\sigma)}{\Gamma(\delta)}\mathcal{I}^{\sigma;\psi}\left[\psi^{\delta-1}(x)(f\circ\psi)^p(x)\right], \end{aligned} \tag{45}$$

and

$$\frac{1}{\Gamma(\delta)}\int_0^x \psi'(y)\left[\psi(x)-\psi(y)\right]^{\delta-1}\psi^{\delta-1}(y)\,g^q\left[\psi(x)-\psi(y)\right]dy$$
$$=\frac{\Gamma(\sigma)}{\Gamma(\sigma)\Gamma(\delta)}\int_0^x \psi'(u)\left[\psi(x)-\psi(u)\right]^{\sigma-1}\psi^{\delta-1}(u)\,(g\circ\psi)^q(u)\,du=\frac{\Gamma(\sigma)}{\Gamma(\delta)}\mathcal{I}^{\sigma;\psi}\left[\psi^{\delta-1}(x)(g\circ\psi)^q(x)\right]. \tag{46}$$

By using (43) and (45), we obtain

$$\begin{aligned}\left[f^p(0)+(f\circ\psi)^p(x)\right]\mathcal{I}^{\delta;\psi}\psi^{\sigma-1}(x) &\leq \frac{\Gamma(\sigma)}{\Gamma(\delta)}\mathcal{I}^{\sigma;\psi}\left[\psi^{\delta-1}(x)(f\circ\psi)^p(x)\right]+\mathcal{I}^{\delta;\psi}\left[\psi^{\sigma-1}(x)(f\circ\psi)^p(x)\right]\\ &\leq 2(f\circ\psi)^p\left(\frac{x}{2}\right)\mathcal{I}^{\delta;\psi}\psi^{\sigma-1}(x),\end{aligned} \tag{47}$$

and using (44) and (46), we get

$$\begin{aligned}\left[g^q(0)+(g\circ\psi)^q(x)\right]\mathcal{I}^{\delta;\psi}\psi^{\sigma-1}(x) &\leq \frac{\Gamma(\sigma)}{\Gamma(\delta)}\mathcal{I}^{\sigma;\psi}\left[\psi^{\delta-1}(x)(g\circ\psi)^q(x)\right]+\mathcal{I}^{\delta;\psi}\left[\psi^{\sigma-1}(x)(g\circ\psi)^q(x)\right]\\ &\leq 2(g\circ\psi)^q\left(\frac{x}{2}\right)\mathcal{I}^{\delta;\psi}\psi^{\sigma-1}(x).\end{aligned} \tag{48}$$

The inequalities (47) and (48) imply that

$$\begin{aligned}&\left[f^p(0)+(f\circ\psi)^p(x)\right]\left[g^q(0)+(g\circ\psi)^q(x)\right]\left(\mathcal{I}^{\delta;\psi}\psi^{\sigma-1}(x)\right)^2\\ &\leq \frac{\Gamma(\sigma)}{\Gamma(\delta)}\mathcal{I}^{\sigma;\psi}\left[\psi^{\delta-1}(x)(f\circ\psi)^p(x)\right]+\mathcal{I}^{\delta;\psi}\left[\psi^{\sigma-1}(x)(f\circ\psi)^p(x)\right]\\ &\quad\times\frac{\Gamma(\sigma)}{\Gamma(\delta)}\mathcal{I}^{\sigma;\psi}\left[\psi^{\delta-1}(x)(g\circ\psi)^q(x)\right]+\mathcal{I}^{\delta;\psi}\left[\psi^{\sigma-1}(x)(g\circ\psi)^q(x)\right].\end{aligned} \tag{49}$$

Similarly as before, we have f and g are positive functions and ψ is increasing function on $[0,\infty)$. Then for any $x>0$, $p\geq 1$, $q\geq 1$, we can write

$$\frac{f^p(0)+(f\circ\psi)^p(x)}{2}\mathcal{I}^{\delta;\psi}\psi^{\sigma-1}(x)\geq 2^{-p}\left[f(0)+(f\circ\psi)(x)\right]^p\mathcal{I}^{\delta;\psi}\psi^{\sigma-1}(x), \tag{50}$$

and

$$\frac{g^q(0)+(g\circ\psi)^q(x)}{2}\mathcal{I}^{\delta;\psi}\psi^{\sigma-1}(x)\geq 2^{-q}\left[g(0)+(g\circ\psi)(x)\right]^q\mathcal{I}^{\delta;\psi}\psi^{\sigma-1}(x). \tag{51}$$

The inequalities (50) and (51) imply that

$$\begin{aligned}&\frac{1}{4}\left[f^p(0)+(f\circ\psi)^p(x)\right]\left[g^q(0)+(g\circ\psi)^q(x)\right]\left(\mathcal{I}^{\delta;\psi}\psi^{\sigma-1}(x)\right)^2\\ &\geq 2^{-p-q}\left[f(0)+(f\circ\psi)(x)\right]^p\left[g(0)+(g\circ\psi)(x)\right]^q\left(\mathcal{I}^{\delta;\psi}\psi^{\sigma-1}(x)\right)^2.\end{aligned} \tag{52}$$

Combining the inequalities (49) and (52), we obtain the desired inequality (40). □

Remark 3. If we put $\psi(x)=x$ for all $x\in[0,\infty)$, then Theorem 7 reduce to Theorem 5 obtained by Dahmani in [15].

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

References

[1] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations* (Vol. 204). elsevier.

[2] Hilfer, R. (2000). *Applications of Fractional Calculus in Physics* (Vol. 35). Singapore, World scientific.

[3] Almeida, R., Malinowska, A. B., & Odzijewicz, T. (2016). Fractional differential equations with dependence on the Caputo–Katugampola derivative. *Journal of Computational and Nonlinear Dynamics, 11*(6), Article No. 061017.

[4] Sousa, J. V. D. C., & de Oliveira, E. C. (2018). On the Ulam–Hyers–Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer operator. *Journal of Fixed Point Theory and Applications, 20*, Artilce No. 96.

[5] Aljaaidi, T.A. & Pachpatte, D.B. (2020). Some Gruss-type inequalities using generalized Katugampola fractional integral. *AIMS Mathematics, 5*(2), 1011-1024.

[6] Aljaaidi, T. A., & Pachpatte, D. B. (2020). The Minkowski's inequalities via ψ-Riemann–Liouville fractional integral operators. *Rendiconti del Circolo Matematico di Palermo Series 2*, 1-14.

[7] Anastassiou, G. A. (2011). *Advances on Fractional Inequalities*. Springer Science & Business Media.

[8] Wang, G., Agarwal, P., & Chand, M. (2014). Certain Grüss type inequalities involving the generalized fractional integral operator. *Journal of Inequalities and Applications, 2014*, Article No. 147.

[9] Hermite, C. (1883). Sur deux limites d'une intégrale définie. *Mathesis, 3*(1), 1-82.

[10] Hadamard, J. (1893). Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *Journal de Mathématiques Pures et Appliquées, 58*, 171-216.

[11] Niculescu, C., & Persson, L. E. (2006). *Convex Functions and their Applications*. Springer, New York.

[12] Dragomir, S. S., & Agarwal, R. P. (1998). Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Applied Mathematics Letters, 11*(5), 91-95.

[13] El Farissi, A., Latreuch, Z., & Belaïdi, B. (2009). Hadamard-type inequalities for twice differentiable functions. *RGMIA Reseaech Report Collection, 12*(1), 7 pages.

[14] Florea, A., & Niculescu, C. P. (2007). A Hermite-Hadamard inequality for convex-concave symmetric functions. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie, 50*(98), 149-156.

[15] Dahmani, Z. (2010). On Minkowski and Hermite-Hadamard integral inequalities via fractional integration. *Annals of Functional Analysis, 1*(1), 51-58.

[16] Sarikaya, M. Z., Set, E., Yaldiz, H., & Basak, N. (2013). Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling, 57*(9-10), 2403-2407.

[17] Dragomir, S.S., Fitzpatrik, S. (1999). The Hadamard's inequality for s -convex functions in the second sense. *Demonstratio Mathematica, 32*(4), 687-696.

[18] Noor, M. A., Noor, K. I., & Awan, M. U. (2015). Some quantum estimates for Hermite–Hadamard inequalities. *Applied Mathematics and Computation, 251*, 675-679.

[19] Liu, W., Wen, W., & Park, J. (2016). Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals. *Journal of Nonlinear Science and Application, 9*(3), 766-777.

[20] Agarwal, P., Jleli, M., & Tomar, M. (2017). Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals. *Journal of Inequalities and Applications, 2017*, Artilce No. 55.

[21] Khan, M. A., Chu, Y. M., Kashuri, A., Liko, R., & Ali, G. (2018). Conformable fractional integrals versions of Hermite-Hadamard inequalities and their generalizations. *Journal of Function Spaces, 2018* Article ID 6928130, 9 pages.

[22] Rostamian Delavar, M., Dragomir, S. S., & De La Sen, M. (2019). A note on characterization of h-convex functions via Hermite-Hadamard type inequality. *Problemy Analiza, 8*(2), 28-36.

[23] Mohammed, P. O. (2019). Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function. *Mathematical Methods in the Applied Sciences, 2019*, 1-11.

[24] Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). *Fractional Integrals and Derivatives* (Vol. 1). Yverdon-les-Bains, Switzerland, Gordon and Breach Science Publishers, Yverdon.

[25] Set, E., Özdemir, M., & Dragomir, S. (2010). On the Hermite-Hadamard inequality and other integral inequalities involving two functions. *Journal of Inequalities and Applications, 2010*, Article No. 148102.