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Flow of a Viscous Fluid Past a Porous Oblate Spheroid at Small Reynolds Number

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

This study deals with the uniform motion of an adhesive, incompressible fluid flowing over a porous oblate spheroid at tiny values of the Reynolds number. These types of problems have been considered by dividing fluid flow into three regions, namely, zone I, zone II, and zone III. In the zone I, which is completely filled with viscous fluid, is the region of the porous oblate spheroid, and in this region fluid flow is governed by the equation suggested by Brinkman. The zone II and the zone III, where the clear fluid flows, are the regions outside the porous oblate spheroid. The fluid flow in these two zones has been discussed using the

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perturbation method given by Proudman and Pearson in which the Stokes stream function is expanded in terms of Reynolds number. This solution is then matched with the Oseen solution, At the interface of zone II and zone I, the matching conditions suggested by Ochoa-Tapia and Whitaker are applied for matching the stream function of the clear fluid region with that of the porous region at the surface of the oblate spheroid. It has been found that the drag on the oblate spheroid reduces with that of the departure from the spherical shape. Similar effects of the drag on the spheroid are obtained when the permeability of the porous medium increases. Also, the drag experienced on the porous oblate spheroid is directly proportional to Reynolds number and the ratio of effective viscosity of the porous medium to the real viscosity of the fluid. The application of a viscous fluid flow past a porous oblate at low Reynolds number is to calculate the friction factor and drag in internal and external flow, hydraulics study, aerofoil design, filtration technology, geothermal energy, and precipitation.

Keywords: Reynolds number; porous oblate spheroid; viscous fluid.

Mathematics Subject Classification (2020): 76S05

1 Introduction

The complication of the flow of fluid in all directions as an obstacle has been considered by Stokes for the motion of the fluid in which the inertial outcomes of the fluid are neglected. Payne and Pell (1960) have discussed the Stokes flow problem around the axially symmetric bodies and obtained the solution by using potential theory developed by Weinstein (1948, 1955). They obtained the streamlet task and drag for the fluid flow over the oval-shaped and oblate spheroids, which agreed with the output of the Oberbeck (1876). Happel and Brenner (1965) have discussed the difficulties of uniform flow over a spheroid. This departs but a tiny bit in shape from a sphere. The polar form of the equation of a spheroid is taken hold as.

$$r^* = c\{1 + \beta_m \vartheta_m(\eta)\}, \tag{1}$$

the coefficients β_m is enough little so that the square and big powers may be eliminated, that is

$$\left(\frac{r^*}{c}\right)^k \approx 1 + k\beta_m \vartheta_m(\eta), \tag{2}$$

here r^* is the distance measured from the centre of the spheroid, c is the polar radius, ϑ_m is Gegenbauer's function of order m and $\eta = \cos\theta$. To satisfy the boundary conditions, they have expressed the product of ϑ_2 and ϑ_m as linear combination of ϑ_{m-2}, ϑ_m and ϑ_{m+2}. The results for an oblate spheroid are deduced by taking $m = 2$ and $\beta_m = 2\in$. All these workers have neglected inertia terms thus taking Reynolds number to be zero.

Aoi (1955) has derived an exact analytical solution of the equations of motion for Oseen's flow past a spheroid by taking the steady state conditions. He has computed the drag experienced by prolate and oblate spheroids for small Reynolds numbers. The study of the flow of the viscous fluid at small Reynolds number about an impervious solid sphere has been investigated by Proudman and Pearson (1957). They have considered the expansions in powers of Reynolds number in two different regions. An inner expansion, which is called Stokes expansion, that is valid in the region that is close to the surface of the sphere, and an external growth, called Oseen's growth or increase in size, which is reasonable for the region that is at the huge interspace from the outside aspect of the sphere. They also assumed that these two different expansions are of the same form of the function, which then are matched using the boundary conditions at a distance far away from the sphere. The results of Proudman and Pearson (1957) for a sphere have been generalized by Breach (1961) for an ellipsoid of revolution, both prolate and oblate. He has given a table for the main constant, which determines the drag, as well as Oseen's solution for various values of eccentricity of prolate and oblate spheroids.

In many situations, the particles that are broadly used in technological progress, particularly in chemical engineering procedures in the industry, are porous. Porous prill is applied considerably in catalytic or motivation reactors. The convectional movement in these porous tiny bits, or atoms, may increase or magnify successfulness components and modify the discrimination of the chemical reactor (see, Nir and Pisman 1977). Porous particles are regularly obtained in the airspace and in additional surrounding structures; here they are obtained by vapor distillation–condensation process (see Pruppacher and Klett 1978). Hence many workers have discussed the flow past porous particles that are spherical or approximately spherical. Feng and Michaelides (1998) considered the problem of fluid flow around a porous sphere for small values. Reynolds number. Following Proudman and Pearson (1957), they assumed that Darcy's law governs the motion of the fluid flow in the region inside the sphere and that in the region that is outside the sphere, Navier-Stokes equations govern the flow of the fluid. They obtained the solution by matching the motion of the fluid inside and outside the porous sphere at the surface of the sphere using the conditions suggested by Saffman's (1971). The problem of fluid flowing inside and past a spheroid that is isolated and permeable has been discussed by Vanishtein et. al. (2002). They assumed that there is no inertia force and took the direction of flow along the axis of the spheroid and obtained the solution of the equation of fluid flowing through the region around the spheroid by applying Stokes equation for creeping flow and that in the region inside the spheroid using Darcy's law. They obtained that the transition of internal fluid flow is very much affected by the shape and dimensions of particles. Srinivasacharya (2003) has studied the creeping flow of adhesive fluid, which passes through a porous estimate sphere neglecting inertia terms. He had concluded the result for an oblate spheroid. Moreover, his calculations for the drag on the imprecise sphere are wrong because the contribution of the term containing $J_4(h)$ in the expansion for the drag should be zero.

For the medium of high porosity, the equation suggested by Brinkman (1947) is so fit for describing the flow of fluids through the permeable medium. Furthermore, for the majority of the flow of fluid in the permeable medium, it is obtained that the coefficient of effectual viscosity m_e is not the same as m, which is the viscosity of clear fluid (see Giveler and Altobeli 1994). When fluid flows in the porous regions, the Brinkman equation is applicable, and for the fluid flows outside the porous regions, the Navier-Stokes equations are used. For the fluid flows at the collaborate of the clear fluid and permeable medium, matching conditions that are suggested theoretically by Ochao–Tapia and Whitaker (1995) and experimentally (Ochao–Tapia and Whitaker 1995) are used. In these matching conditions, they assumed that at the interface there is a continuousness of velocity and the normal stress while shearing stresses are discontinuous at the interface. Using these assumptions, the problem of viscous fluid flowing over a permeable, spherical-shaped structure has been studied by Srivastava and Srivastava (2005) for small values of the Reynolds number. In this problem, the results of Srivastava and Srivastava (2005) are used to discuss the flow past a porous oblate spheroid by applying the method suggested by Happel and Brenner (1965) for satisfying the matching conditions at the interface of the spheroid. Combined boundary layer complications of this type have been obtained by many workers, like Neale et. al. (1973), Adler (1981), Jones (1973), Srivastava (1999), Langlois (1964), Iyengar & Radhika (2015), and Alexander et. al. (2022).

The effects of radiation and thermal diffusion on MHD heat transfer flow of a dusty viscoelastic fluid between two moving parallel plates have been studied by B. M. Reddy et. al. (2018). The radiative MHD Walter's Liquid-B Flow Past a Semi-Infinite Vertical Plate in the Presence of Viscous Dissipation with a Heat Source, Engineering Transactions, has been presented by Chenna et. al. (2021). An analytical study on induced magnetic field with radiating fluid over a porous vertical plate with heat generation has been studied by Chenna et. al. (2017). Radiation effect on MHD oscillatory flow in a channel filled through a porous medium with heat generation, Journal of Mathematical Control Science and Applications Chemical reaction and radiation absorption effects on convective flows past a porous vertical wavy channel with travelling thermal waves, International Journal of Fluid Mechanics Research; MHD effect on convective flow of dusty viscous fluid with fraction in a porous medium and heat generation and magneto-compound reaction of convective flow via a porous inclined plate with Heat Energy Absorption has been studied by Chenna et al. (2019, 2020, 2021, 2024). Reddy et. al. (2021) has been studied hall effect on MHD flow of a visco-elastic fluid through porous medium over an infinite vertical porous plate with Heat Source. G. Balreddy et. al. (2024) has been proposed radiation absorption and chemical reaction effects on MHD flow through porous medium past an exponentially accelerated inclined plate with variable temperature.

The effects of Hall current and rotation, heat generation on MHD free convection heat and mass transfer flow past an accelerated vertical plate, and analytical solutions for transient free convection MHD flow through a porous medium between two vertical plates with a heat source have been studied by G. Balreddy et. al. (2023, 2022). Ramesh Babu et. al. (2022) has been studied chemical reaction and Hall effects on unsteady flow past an isothermal vertical plate in a rotating fluid with variable mass diffusion with a heat source. In another paper, Ramesh Babu et. al.(2024) have studied variable temperature, radiation absorption, and chemical reaction effects on unsteady MHD flow through a porous medium past an oscillating inclined plate. Ravikumar et. al. (2023) has studied the effects of diffusion on the mechanism of peristaltic flow at slip boundaries when internal Joule heating is present. In another paper, Ravikumar et. al. (2023) has studied the effects of Joule heating and reaction mechanisms on couple stress fluid flow with peristalsis in the presence of a porous material through an inclined channel. The significance of heat and mass transport in peristaltic flow of Jeffrey material subject to chemical reaction and radiation phenomenon through a tapered channel and the study of hall current, radiation, and velocity slip on hydro magnetic physiological hemodynamic fluid with porous medium through joule heating and mass transfer in the presence of chemical reaction has been presented by Ravikumar et. al.(2022, 2018). He found that the temperature of the fluid rises with the lead of Hartmann number (M), porosity parameter (Da), radiation parameter (N), Prandtl number (Pr), Brinkman number (Br), heat source parameter (v), and Hall current parameter (m). he has also obtained that the concentration profile lacks with the lead in chemical reaction parameters.

Ravikumar and Abzal (2017) have presented the combined influence of hall currents and joule heating on hemodynamic peristaltic flow with porous medium through a vertical tapered asymmetric channel with radiation. They have obtained the pressure gradient lacks by rise in hall current parameter, porosity parameter, and volumetric flow rate. The temperature of the fluid leads by lead of magnetic parameter M, parameter N, parameter Pr, and Br, and lacks by leads in m and Da. In another paper, Ravi Kumar (2023) has studied the rotation effect on a fluid model exhibiting thermo-diffusion in a porous environment subject to convective boundary conditions through a slanted conduit. Khan et. al.(2023) have presented the understanding of Prandtl fluid flow in conduits with slip boundary conditions: Implications for engineering and physiology. Ravi Kumar (2015) has studied the effect of couple stress fluid flow on magneto hydrodynamic peristaltic blood flow with a porous medium through an inclined channel in the presence of a slip effect.

2 Formulation of the Problem

Let us consider the flow of viscous fluid past over a porous oblate spheroid with an uniform velocity U parallel to its axis of revolution.

The spheroid is completely filled with the fluid. Let (r^*, θ, ϕ) be the spherical polar coordinates with centre of the spheroid being at the origin, then the polar equation of the spheroid is

$$r^* = a(1 - \in \cos^2\theta) = a[1 - \in + 2 \in \vartheta_2(\eta)] = c\,[1 + 2 \in \vartheta_2(\eta)], \tag{3}$$

where $a = c(1 + \in)$ and $\in$ is a very small quantity whose square and higher powers are negligible. The constants a and c are called equatorial radius and polar radius of the spheroid respectively. In this problem the region of fluid flow has been divided in three zones (see Fig. 1). For the fluid flow inside the porous oblate spheroid, region is taken as Zone I and following Brinkman equations[3] governs the flow of fluid.

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mu_e \nabla^2 \mathbf{v} - \frac{\mu \mathbf{v}}{k}, \tag{4}$$

here k denotes the permeability of porous oblate spheroid, $\mathbf{v}$ denotes the velocity vector and p denotes the pressure at any point in porous oblate spheroid. The parameter μ_e represents the effectual viscosity of the porous oblate spheroid. It is assumed to be dissimilar from parameter μ. This represents the coefficient of viscosity in clear fluid zone. The regions where clear fluid flows are Zones II and III and the flow fluid. This zone is controlled by the Navier–Stokes equation. For the flow of clear fluid near the surface of the porous oblate spheroid in Zone II, Stokes' approximations are reasonable.

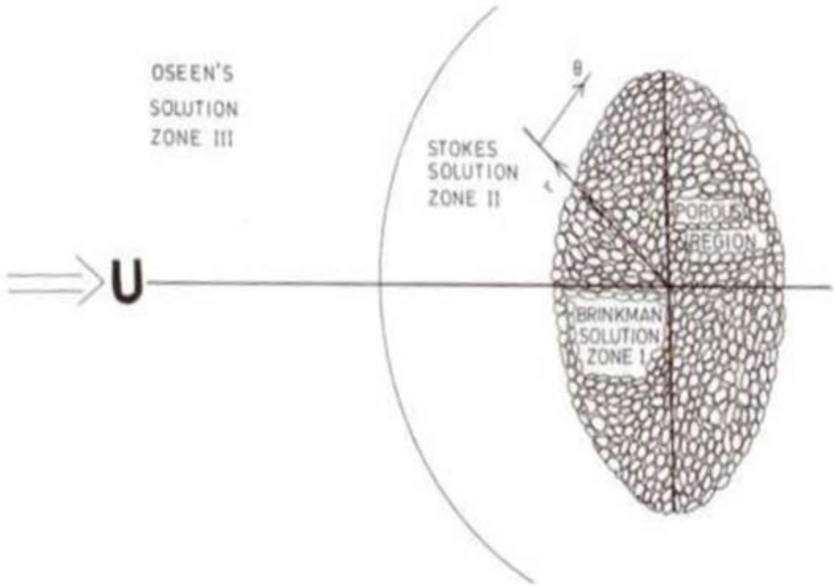

Fig. 1. The schematics diagram of the problem

Let us assume that u and v be velocity of fluids in the direction of r^* and θ respectively. Then Stokes stream function Ψ in spherical polar coordinates system is represented by

$$u = \frac{1}{(r^*)^2 \sin\theta}\frac{\partial\Psi}{\partial\theta}, \qquad v = -\frac{1}{r^* \sin\theta}\frac{\partial\Psi}{\partial r^*}, \tag{5}$$

Assuming that the sign i denotes the zone undergoing assumption. The boundary situation for the flow issue can be represented as given below:

$$u^{(1)} \text{ and } v^{(1)} \text{ are finite at } r^* = 0, \tag{6}$$

$$u^{(3)} \to U\cos\theta, \quad v^{(3)} \to -U\sin\theta, \quad \text{as } r^* \to \infty, \tag{7}$$

In the interface of porous oblate spheroid and clear fluid $r^* = c[1 + 2\in \vartheta_2(\eta)]$. We consider that the components of velocity along with normal stress $\tau_{r^*r^*}$ are continual and shearing stress $\tau_{r^*\theta}$ has jump which is specified by the equation has been proposed by (Ochoa–Tapia and Whitaker 1995). In this notations, at the interface $r^* = c[1 + 2\in\vartheta_2(\eta)]$, these conditions are given by

$$\Psi^{(1)} = \Psi^{(2)}, \tag{8}$$

$$\frac{\partial\Psi^{(1)}}{\partial r} = \frac{\partial\Psi^{(2)}}{\partial r}, \tag{9}$$

$$\tau^{(1)}_{r^*r^*} = \tau^{(2)}_{r^*r^*}, \tag{10}$$

$$\tau^{(1)}_{r^*\theta} - \tau^{(2)}_{r^*\theta} = \frac{\beta\mu}{\sqrt{k}} v^{(1)}, \tag{11}$$

where β represents a constant of order. The sign of β is either positive or negative. The formulations for $\tau_{r^*\theta}$ and $\tau_{r^*r^*}$ in the spherical polar coordinates system is represented by

$$\tau_{r^*\theta} = \mu\left(\frac{1}{r^*}\frac{\partial u}{\partial \theta} - \frac{v}{r^*} + \frac{\partial v}{\partial r^*}\right), \tag{12}$$

$$\tau_{r^*r^*} = -p + 2\mu\frac{\partial u}{\partial r^*} , \tag{13}$$

3 Solution of Equations

Since equations of Brinkman and Stokes are alike same. Let us consider the following variables for zones I and II as

$$\Psi = c^2 U \Psi^{(i)}, \qquad p = \frac{\mu U}{c} p^{(i)} \quad , \quad \text{for} \quad i = 1, 2 \text{ and } r^* = c\, r, \tag{14}$$

In zone I, using the above variables, eq. (4) becomes

$$\gamma^2 E^4 \Psi^{(1)} - \sigma^2 E^2 \Psi^{(1)} = 0, \tag{15}$$

where $\sigma = \frac{c}{\sqrt{k}}$ represents the Darcy number and the operator E^2 in dimensionless form is defined as

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\eta^2}{r^2}\frac{\partial^2}{\partial \eta^2}, \tag{16}$$

In terms of $\Psi^{(2)}$, for the zone II, Navier–Stokes equation can be written as:

$$\frac{1}{r^2}\frac{\partial(\psi^{(2)}, E^2\psi^{(2)})}{\partial(r,\eta)} + \frac{2E^2\psi^{(2)}}{r^2}\left[\frac{\eta}{1-\eta^2}\frac{\partial \psi^{(2)}}{\partial r} + \frac{1}{r}\frac{\partial \psi^{(2)}}{\partial \eta}\right] = \frac{1}{\mathrm{Re}} E^4 \psi^{(2)} , \tag{17}$$

where Re = Uc/υ represents the Reynolds number and $\upsilon = \mu/\rho$ denotes the kinematic coefficient of viscosity. The following Oseen's variable is introduced for the flow in zone III

$$\xi = (\mathrm{Re})r = \mathrm{Re}\frac{r^*}{c} = \frac{Ur^*}{\upsilon} , \tag{18}$$

$$\psi^{(3)} = \mathrm{Re}^2 \psi^{(2)} = \frac{\mathrm{Re}^2 \psi}{c^2 U} = \frac{U}{(\upsilon^2)}\psi , \tag{19}$$

In zone III, the following expression for $\Psi^{(3)}(r, \eta)$ can be obtained using Navier–Stokes equation (see, Langlois 1964),Page (148), eq. (4.7)]

$$\Psi^{(3)} = \frac{1}{2}\left[\left\{\frac{c}{2r^*} + \left(\frac{r^*}{c}\right)^2\right\}(1-\eta^2) - \frac{4B}{\mathrm{Re}}(1+\eta)(1-e^{-(\mathrm{Re}\, r^*/2c)(1-\eta)})\right], \tag{20}$$

This formulations of $\Psi^{(3)}(r^*, \eta)$ represents the solution of the equation (17). If we take (B/12) instead of B in the equation (20) then with proper adjustment of variables we get the Ψ proposed by Breach as the Oseen's solution for spheroids (see, equation (14), page 307 of Breach 1961), Hence it is justified to take (20) as Oseen's solution for the flow over a porous oblate spheroid. Value of B is found to be of the form $B_1 + \in B_2$ when it is evaluated such that (20) matched with the expression of Ψ for Stokes solution at the surface of spheroid given by (3) which contains $\in$. It has been found that when the oblate porous spheroid is taken in place of the solid impervious sphere of radius 'c' the constant B = 3/4 and (20) represents a right answer of Oseen's formulation (see, Langlois 1964). Putting Oseen's variable in solution (20) and expressing in powers of Re. we obtain the expression given below

$$\psi^{(3)} = (1/2)\zeta^2(1-\eta^2) - B\,\mathrm{Re}[\zeta(1-\eta^2) - (1/4)\zeta^2(1-\eta^2)(1-\eta) + 0(\zeta^3)], \tag{21}$$

It may be noted that if in the equation (21) we want to write $\overline{\mathrm{R}}\mathrm{e} = aU/\upsilon$ instead of Re [Re = $(1-\in)\ \overline{\mathrm{R}}\mathrm{e}$], then the constant B should be taken as $B(1-\in)$. In order to match this stream function given by (21) with the Stokes stream function $\psi^{(2)}$, we revise it in the Stokes' variable as:

$$(\mathrm{Re})^{-2}\psi^{(3)} = (1/2)(r^2 - 2Br)(1-\eta^2) + (B/4)\,\mathrm{Re}\,r^2(1-\eta^2)(1-\eta) + 0(\mathrm{Re}^2), \tag{22}$$

In order to solve equations (4) and (17) respectively in zones I and II, we take the following expressions of streamlet function $\psi^{(i)}$ and pressure $p^{(i)}$ in terms of Reynolds number Re is represented as:

$$\psi^{(i)} = \psi_0^{(i)} + \mathrm{Re}\,\psi_1^{(i)} + 0(\mathrm{Re}^2), \qquad i = 1, 2 \tag{23}$$

$$P^{(i)} = P_0^{(i)} + \mathrm{Re}\,P_1^{(i)} + 0(\mathrm{Re}^2), \qquad i = 1, 2 \tag{24}$$

3.1 First approximation

Substituting (23) in the equations (15) and (17), we get the following differential equations for $\psi_0^{(1)}$ and $\psi_0^{(2)}$ on equating the terms on both sides which are independent of Re

$$D^4\psi_0^{(1)} - \alpha^2 D^2\psi_0^{(1)} = 0, \tag{25}$$

$$D^4\psi_0^{(2)} = 0, \tag{26}$$

where $\alpha = \sigma/\gamma$. The solution of (25) is an infinite series of the type $\sum_{n=2}^{\infty} F_{0\eta}(r)\vartheta_n(\eta)$ (see, equation (4 – 25.3) of (Happel and Brenner 1965). Here, we are discussing the flow past porous spheroid given by equation (3) in which only $\vartheta_2(\eta)$ occurs. In the matching conditions for flow of fluid at the interface of the oblate porous spheroid and free flow zone only $\vartheta_2^2(\eta)$ occurs which can be expressed as

$$\vartheta_2^2(\eta) = \frac{2}{5}[\vartheta_2(\eta) - \vartheta_4(\eta)], \tag{27}$$

Hence, we take only two terms of the series and assume the following form for $\psi_0^{(1)}$ and $\psi_0^{(2)}$

$$\psi_0^{(1)} = 2F_{02}(r)\vartheta_2(\eta) + \in F_{04}(r)\vartheta_4(\eta), \tag{28}$$

$$\psi_0^{(2)} = 2f_{02}(r)\vartheta_2(\eta) + \in f_{04}(r)\vartheta_4(\eta), \tag{29}$$

Substituting these expressions for $\psi_0^{(1)}$ and $\psi_0^{(2)}$ in the equations (25) and (26) respectively, we obtain the differential equations for $F_{02}(r)$, $F_{04}(r)$, $f_{02}(r)$ and $f_{04}(r)$ which are integrated with respect to r. Then we have obtained the following equations for $F_{02}(r)$, $F_{04}(r)$, $f_{02}(r)$ and $f_{04}(r)$ respectively.

$$F_{02}(r) = Kr^2 + c\left\{\frac{\sinh\alpha r}{\mathrm{r}} - \alpha\cosh\alpha r\right\}, \tag{30}$$

$$F_{04}(r) = K_{04}r^4 + c_{04}\left\{\frac{6\alpha^2 r^2 + 15}{r^3}\sinh\ \alpha r - \frac{\alpha(\alpha^2 r^2 + 15)}{r^2}\cosh\ \alpha r\right\}, \tag{31}$$

$$f_{02}(r) = \frac{A}{r} - Br + \frac{1}{2}r^2, \tag{32}$$

$$f_{04}(r) = \frac{A_{04}}{r^3} + \frac{B_{04}}{r}. \tag{33}$$

In writing the expressions (30)–(33), the constants of integration which are multipliers of the solutions which are not defined at the centre of the oblate porous spheroid or at the large distance from the surface are assumed to be zero.

For the region given by zone II when $r \to \infty$, we assume take the following first approximate solution as

$$\psi_0^{(2)} = \frac{1}{2}(r^2 - 2Br)(1 - \eta^2), \tag{34}$$

which is then matched with the first term that is not dependent of Re of the result given in (22). The other constants A, B, K, C, A_{04}, B_{04}, K_{04} and C_{04} are to be found by matching the solutions (32) and (33) with that of equations of Brinkman (30) and (31) at the surface of oblate porous spheroid.

In the matching conditions at the surface of spheroid, we take $r = 1 + 2\in\vartheta_2(\eta)$ in $F_{02}(r)$ and $f_{02}(r)$ and $r = 1$ in $F_{04}(r)$ and $f_{04}(r)$ because terms containing $\in^2$ and higher powers of $\in$ are neglected. We expand $F_{02}(r)$ and $f_{02}(r)$ in powers of $\in$ (neglecting $\in^2$ and higher powers) at the interface i.e. we take

$$F_{02}(1 + 2\in\vartheta_2(\eta)) = F_{02}(r) + 2\in F'_{02}(1)\vartheta_2(\eta) + 0(\in^2), \tag{35}$$

$$f_{02}(1 + 2\in\vartheta_2(\eta)) = f_{02}(1) + 2\in f'_{02}(1)\vartheta_2(\eta) + 0(\in^2), \tag{36}$$

Substituting (28), (29) in the matching condition (8) and using (35) and (36), we get

$$2\{F_{02}(1) + 2\in F'_{02}(1)\vartheta_2(\eta)\}\vartheta_2(\eta) + \in F_{04}(1)\vartheta_4(\eta) = 2\{f_{02}(1) + 2f'_{02}(1)\vartheta_2(\eta)\} + \in f_{04}(1)\vartheta_4(\eta), \tag{37}$$

Substituting equation (27) and the equations (30)–(33) in the equation (37), we get

$$2[K + C(\sinh\alpha - \alpha\cosh\alpha) - (A - B + 1/2)]\vartheta_2(\eta) + \frac{8\in}{5}[2K - C\{(\alpha^2 + 1)\sinh\alpha$$
$$- \alpha\cosh\ \alpha) + (A + B - 1)](\vartheta_2(\eta) - \vartheta_4(\eta)) + \in[K_{04} + C_{04}\{(6\alpha^2 + 15)\sinh\alpha$$
$$- \alpha(\alpha^2 + 15)\cosh\alpha\} - (A_{04} + B_{04})]\vartheta_4(\eta) = 0, \tag{38}$$

This equation suggests that the constants A, B, C and K should be taken of the following form

$$A = A_{01} + \in A_{02},\quad B = B_{01} + \in B_{02},\quad C = C_{01} + \in C_{02},\quad K = K_{01} + \in K_{02}, \tag{39}$$

Substituting equation (39) in equation (38) and then equating the coefficient of $\vartheta_2(\eta)$, $\in\vartheta_2(\eta)$ and $\in\vartheta_4(\eta)$ on both sides, we have the following three equations:

$$K_{01} + C_{01}(\sinh\alpha - \alpha\cosh\alpha) - A_{01} + B_{01} = 1/2, \tag{40}$$

$$K_{02} + C_{02}(\sinh\alpha - \alpha\cosh\alpha) - A_{02} + B_{02} = \frac{4}{5}[-2K_{01} + C_{01}\{(\alpha^2+1)\sinh\alpha - \alpha\cosh\alpha\}] - \frac{4}{5}[A_{01} + B_{01} - 1], \tag{41}$$

$$K_{04} + C_{04}\{(6\alpha^2 + 15)\sinh\alpha - \alpha(\alpha^2 + 15)\cosh\alpha\} - A_{04} - B_{04} = \frac{8}{5}[2K_{01} - C_{01}\{(\alpha^2+1)\sinh\alpha - \alpha\cosh\alpha\} + A_{01} + B_{01} - 1]. \tag{42}$$

Substituting equations (28), (29) in the matching condition (9), using equations (27), (30)–(33), (35), (36) and (39) and following the same procedure as above, we get the another set of equations for (A_{01}, B_{01}, C_{01}, K_{01}), (A_{04}, B_{04}, C_{04}, K_{04}):

$$2K_{01} + C_{01}\{\alpha\cosh\alpha - (\alpha^2 + 1)\sinh\alpha\} + A_{01} + B_{01} = 1, \tag{43}$$

$$2K_{02} + C_{02}(\alpha\cosh\alpha - (\alpha^2 + 1)\sinh\alpha) + A_{02} + B_{02} = -\frac{4}{5}[-2K_{01} + C_{01}\{(\alpha^2+1)\sinh\alpha - \alpha(\alpha^2+1)\cosh\alpha\}] + \frac{4}{5}[2A_{01} + 1], \tag{44}$$

$$4K_{04} - C_{04}\{(\alpha^4 + 21\alpha^2 + 45)\sinh\alpha - \alpha(6\alpha^2 + 45)\cosh\alpha\} + 3A_{04} + B_{04} = -\frac{8}{5}[2K_{01} + C_{01}\{(\alpha^2+2)\sinh\alpha - \alpha(\alpha^2+2)\cosh\alpha\}] - \frac{8}{5}[2A_{01} + 1] \tag{45}$$

Equating the velocity components on two sides of the interface we have obtained six conditions (40) – (45) from (8) and (9). Now, we shall derive similar equations from (10) and (11) given in the stress components. As τ_{rr} contains pressure, we shall derive the expressions for $P_0^{(1)}$ and $P_0^{(2)}$. Writing equation (4) and Navier–Stokes equation in terms of spherical polar coordinates and substituting $\psi_0^{(i)}$ and then the resulting differential equation is integrated, which gives the expressions of pressure for the first estimation or approximation in regions I and II respectively as

$$P_0^{(1)} = \left[\gamma^2\left\{F_{02}''' - 2\frac{F_{02}'}{r^2} + \frac{4}{r^3}F_{02}\right\} - \sigma^2 F_{02}'(r)\right]\eta, \tag{46}$$

$$P_0^{(2)} = \left[f_{02}''' - 2\frac{f_{02}'}{r^2} + \frac{4}{r^3}f_{02}\right]\eta, \tag{47}$$

Substituting equations (13), (5), (27), (28)–(33) and (39) in the matching condition (10) and proceeding as earlier we obtain the following equations:

$$\begin{aligned} &C_{01}\gamma^2\{(\alpha^3 - 12\alpha)\cosh\alpha - (\alpha^4 - 3\alpha^2 - 12)\sinh\alpha\} \\ &-\sigma^2[2K_{01} - C_{01}\{(\alpha^2+1)\sinh\alpha - \alpha\cosh\alpha\}] = 6(2A_{01} - B_{01}), \end{aligned} \tag{48}$$

$$\begin{aligned} &C_{02}\gamma^2\{(\alpha^3 - 12\alpha)\cosh\alpha - (\alpha^4 - 3\alpha^2 - 12)\sinh\alpha\} \\ &-\sigma^2[2K_{02} - C_{02}\{(\alpha^2+1)\sinh\alpha - \alpha\cosh\alpha\}] - (12A_{02} - 6B_{02}) \end{aligned}$$

$$+\frac{2}{5}C_{01}\gamma^2\{-(\alpha^4-18\alpha^2-48)\sinh\alpha+(\alpha^5-2\alpha^3-48\alpha)\cosh\alpha\}$$
$$=\frac{2}{5}\sigma^2[2K_{01}+C_{01}\{(\alpha^2+2)\sinh\alpha-(2\alpha+\alpha^3)\cosh\alpha\}]+\frac{2}{5}[-48A_{01}+12B_{01}]\ , \tag{49}$$

$$\gamma^2[2K_{04}-C_{04}\{(\alpha^2+33\alpha^2+75)\sinh\ \alpha-(8\alpha^3+75)\cosh\ \alpha\}]+5A_{04}+3B_{04}$$
$$+\frac{2}{5}C_{01}\gamma^2\{-(\alpha^4-18\alpha^2-48)\sinh\alpha+(\alpha^5-2\alpha^3-48\alpha)\cosh\alpha\}$$
$$=\frac{2}{5}\sigma^2[2K_{01}+C_{01}\{(\alpha^2+2)\sinh\alpha-(2\alpha+\alpha^3)\cosh\alpha\}]+\frac{2}{5}[-48A_{01}+12B_{01}], \tag{50}$$

In the same way, substituting equations (12), (5), (27), (28)–(33) and (39) in the matching condition (11) and proceeding as earlier, we get

$$\gamma^2[C_{01}\{(3\alpha^2+6)\sinh\ \alpha-\alpha(6+\alpha^2)\cosh\ \alpha\}]-6A_{01}=\beta\sigma(1-A_{01}-B_{01})\ , \tag{51}$$

$$\gamma^2[C_{02}\{(3\alpha^2+6)\sinh\ \alpha-\alpha(6+\alpha^2)\cosh\ \alpha\}]-6A_{02}+\beta\sigma(A_{02}+B_{02})$$
$$=\frac{4}{5}[\gamma^2C_{01}\{(\alpha^4-9\alpha^2+18)\sinh\alpha-(3\alpha^2-18\alpha)\cosh\alpha-18A_{01}]+\frac{4}{5}\beta\sigma[2A_{01}+1)\ , \tag{52}$$

$$\gamma^2[16K_{04}+C_{04}\{(8\alpha^4+201\alpha^2+450)\sinh\ \alpha-(\alpha^5+51\alpha^3+450\alpha)\cosh\ \alpha\}]$$
$$-(30A_{04}+16B_{04})+\beta\sigma(3A_{04}+B_{04})$$
$$=\frac{8}{5}[\gamma^2C_{01}\{-(\alpha^4-9\alpha^2+18)\sinh\alpha-(3\alpha^2-18\alpha)\cosh\alpha\}-18A_{01}]\ -\frac{8}{5}\beta\sigma[2A_{01}+1)\ , \tag{53}$$

For given α, γ and β , twelve constants (A_{01}, B_{01}, C_{01}, K_{01}), (A_{02}, B_{02}, C_{02}, K_{02}) and (A_{04}, B_{04}, C_{04}, K_{04}) can be calculated from equations (40)–(45) and equations (48)–(53). In Table 1, we have calculated these constants for various values of γ by taking σ = 5.0 and β = 0.5, – 0.5. In order to study the effects of variability of permeability parameter of the oblate porous spheroid, we determined above constant for γ^2 = 1.0, β = 0.5, –0.5 and σ = 5.0, 6.0, 7.0, 8.0, 9.0, 10.0 and are given in Table 2.

3.2 Second approximation

In this approximation, first we determine the solution in zone II near the plane or surface of the porous oblate spheroid, which is obtained by matching this solution with the corresponding solution for the flow in zone III. Then for the flow in zone I, the solution for the second approximation is obtained by matching it with that of the corresponding solutions of zone II. Using equations (23), (29), (32), and (33) in equation (17), the following differential equation for $\psi_1^{(2)}(r,\eta)$ is obtained when the coefficient of Reynolds Number Re is equated on both sides of equation. The function for $\psi_1^{(2)}(r,\eta)$ gives the second approximation for Stokes stream function in the zone II:

$$E^4\psi_1^{(2)}=-12B\left(\frac{2A}{r^5}-\frac{2B}{r^3}+\frac{1}{r^2}\right)\vartheta_2(\eta)+\left[\left\{B_{04}\left(\frac{11}{84}+\frac{1}{21r}\right)\right.\right.$$
$$\left.+\frac{25}{84r^2}A_{04}\right\}\vartheta_3(\eta)-\frac{40}{77}\left\{B_{04}\left(\frac{5}{4}+\frac{1}{r}\right)+\frac{7}{4r^2}A_{04}\right\}\vartheta_4(\eta)$$
$$\left.+\frac{5}{7}\left\{B_{04}\left(\frac{3}{2}+\frac{1}{r}\right)+\frac{5}{2r^2}A_{04}\right\}\vartheta_5(\eta)+\frac{25}{33}\left\{B_{04}\left(\frac{5}{4}+\frac{1}{r}\right)+\frac{7}{4r^2}A_{04}\right\}\vartheta_7(\eta)\right]\left(\frac{20B_{04}}{r^7}\right)\in\ , \tag{54}$$

In writing the equation (54) when $\psi_0^{(2)}(\eta)$ is substituted in left hand side of equation (17), the derivatives of $\vartheta_2(\eta),\vartheta_4(\eta)$ and their products appear which are rearranged in terms of Gagenbaur functions $\vartheta_3(\eta),\vartheta_4(\eta)$, $\vartheta_5(\eta),\vartheta_7(\eta)$. Following equation (54), the following form for $\psi_1^{(2)}$ is assumed:

$$\psi_1^{(2)} = 2\, f_{12}(r)\vartheta_2(\eta) + 2\, f_{13}(r)\vartheta_3(\eta) + \in [\, f_{14}(r)\vartheta_4(\eta) + f_{15}(r)\vartheta_5(\eta) + f_{17}(r)\vartheta_7(\eta)]\,, \tag{55}$$

This form of $\psi_1^{(2)}(r,\eta)$ is taken according to the coefficient of Reynolds Number Re in the equations for $\psi^{(3)}(r, \eta)$. Putting equation (55) in (54) and equating the coefficients of $\vartheta_2(\eta)$, $\vartheta_3(\eta)$, $\vartheta_4(\eta)$, $\vartheta_5(\eta),\vartheta_7(\eta)$ on both sides of (54), we get

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2 f_{12}(r) = 0\,, \tag{56}$$

$$\left(\frac{d^2}{dr^2} - \frac{6}{r^2}\right)^2 f_{13}(r) = -12B\left(\frac{2A}{r^5} - \frac{2B}{r^3} + \frac{1}{r^2}\right) + 20 \in B_{04}\left[B_{04}\left(\frac{11}{84r^7} + \frac{1}{21r^8}\right) + \frac{25}{84r^9}A_4\right], \tag{57}$$

$$\left(\frac{d^2}{dr^2} - \frac{12}{r^2}\right)^2 f_{14}(r) = -\frac{800}{77}B_{04}\left[B_{04}\left(\frac{5}{4r^7} + \frac{1}{r^8}\right) + \frac{7}{4r^9}A_{04}\right], \tag{58}$$

$$\left(\frac{d^2}{dr^2} - \frac{20}{r^2}\right)^2 f_{15}(r) = \frac{100B_{04}}{7}\left[B_{04}\left(\frac{3}{2r^7} + \frac{1}{r^8}\right) + \frac{5}{2r^9}A_{04}\right], \tag{59}$$

$$\left(\frac{d^2}{dr^2} - \frac{42}{r^2}\right)^2 f_{17}(r) = \frac{500}{33}B_{04}\left[B_{04}\left(\frac{5}{4r^7} + \frac{1}{r^8}\right) + \frac{7}{4r^9}A_{04}\right]. \tag{60}$$

When large values of r are taken, the solution $f_{12}(r)$ of the equation (56) should be matched with the coefficients of $\mathrm{Re}(1 - \eta^2)$ in $\psi^{(3)}(r, \eta)$ which given in the equation (22). Thus, integrating (56), the solution for $f_{12}(r)$ is given by

$$f_{12}(r) = \frac{B}{2}\left(\frac{A}{r} - Br + \frac{1}{2}r^2\right) = \frac{B}{2}f_{02}(r) \tag{61}$$

Integrating equations (57)–(60), the expressions for $f_{13}(r)$, $f_{14}(r)$, $f_{15}(r)$ and $f_{17}(r)$ are given by

$$f_{13}(r) = \frac{M}{r^2} + N + \frac{B}{4}\left(\frac{2A}{r} + 2Br - r^2\right) + \in B_{04}\left[\left(\frac{55}{3024\,r^3} + \frac{5}{2646\,r^4}\right)B_{04} + \frac{5}{1008\,r^5}A_{04}\right], \tag{62}$$

$$f_{14}(r) = \frac{A_{14}}{r^3} + \frac{B_{14}}{r} - \in B_{04}\left[\frac{10}{231r^4}B_{04} + \frac{25}{7623r^5}A_{04}\right], \tag{63}$$

$$f_{15}(r) = \frac{A_{15}}{r^4} + \frac{B_{15}}{r^2} + \in B_{04}\left[\frac{15}{16r^3}B_{04} + \frac{25}{252r^3}A_{04}\right], \tag{64}$$

$$f_{17}(r)=\frac{A_{17}}{r^6}+\frac{B_{17}}{r^4}+\in B_{04}\left[\frac{125}{2376r^3}B_{04}-\frac{125}{792r^5}A_{04}\right]. \tag{65}$$

where M, N, A_{14}, B_{14}, A_{15}, B_{15}, A_{17}, B_{17} are constants of integration. These constants are to be determined under the condition that the expression for $\psi_1^{(2)}$ given in (55) matches with the corresponding solution of Brinkman problem for flow in zone I. It may be found that when $f_{13}(\eta)$, $f_{14}(\eta)$, $f_{15}(\eta)$ and $f_{17}(\eta)$ are substituted in $\psi_1^{(2)}(r,\eta)$, we get the following expression of $\psi_1^{(2)}(r,\eta)$ for large values for r

$$\psi_1^{(2)}(r,\eta)=\frac{B}{4}r^2(1-\eta^2)(1-\eta)$$

which is exactly similar to that of the expression of Oseen's solution designated by the coefficient of Re in the equation (22). Now, we shall determine the solution for the second approximation of the Brinkman equation in zone I. The flow of fluid in zone I is not dependent of Re so when $\psi^{(1)}$ is expanded in powers of Re and substituted in (15), the function $\psi_1^{(1)}$ which is coefficient of Re satisfies the following equation

$$\gamma^2E^4\psi_1^{(1)}-\sigma^2E^2\psi_1^{(1)}=0\,, \tag{66}$$

The expression for $\psi_1^{(1)}$ has to match with that of $\psi_1^{(2)}(r,\eta)$ given in the equation (55) at

$r = 1 + 2\in\vartheta_2(\eta)$. Hence we assume the following form for $\psi_1^{(1)}(r,\eta)$:

$$\psi_1^{(1)}(r,\eta)=2F_{12}(r)\vartheta_2(\eta)+2F_{13}(r)\vartheta_3(\eta)+\in[F_{14}(r)\vartheta_4(\eta)+F_{15}(r)\vartheta_5(\eta)+F_{17}(r)\vartheta_7(\eta)]\,, \tag{67}$$

When the expression of $\psi_1^{(1)}(r,\eta)$ given in equation (67) is substituted in equation (66), the differential equation obtained for $F_{12}(r)$ is same as the differential equation for $F_{02}(r)$ and it has to match with $\psi_1^{(2)}$ at $r = 1 + 2\in\vartheta_2(\eta)$, hence $F_{12}(r)$ is given by

$$F_{12}(r)=\frac{B}{2}F_{02}(r)\,. \tag{68}$$

The differential equations satisfied by $F_{13}(r)$, $F_{14}(r)$, $F_{15}(r)$ and $F_{17}(r)$ are given by

$$\left(\frac{d^2}{dr^2}-\frac{6}{r^2}\right)\left(\frac{d^2}{dr^2}-\frac{6}{r^2}-\alpha^2\right)F_{13}(r)=0\,, \tag{69}$$

$$\left(\frac{d^2}{dr^2}-\frac{12}{\alpha}\right)\left(\frac{d^2}{dr^2}-\frac{12}{r^2}-\alpha^2\right)F_{14}(r)=0\,, \tag{70}$$

$$\left(\frac{d^2}{dr^2}-\frac{20}{r^2}\right)\left(\frac{d^2}{dr^2}-\frac{20}{r^2}-\alpha^2\right)F_{15}(r)=0\,, \tag{71}$$

$$\left(\frac{d^2}{dr^2}-\frac{42}{r^2}\right)\left(\frac{d^2}{dr^2}-\frac{42}{r^2}-\alpha^2\right)F_{17}(r)=0\,. \tag{72}$$

Integrating equations (69) – (72), we obtain the following solutions:

$$F_{13}(r) = Sr^3 + T\left\{\frac{3+\alpha^2 r^2}{r^2}\sinh\ \alpha r - \frac{3\alpha}{r}\cosh\ \alpha r\right\}, \tag{73}$$

$$F_{14}(r) = K_{14}r^4 + C_{14}\left\{\frac{6\alpha^2 r^2 + 15}{r^3}\sinh\ \alpha r - \frac{\alpha(\alpha^2 r^2 + 15)}{r^2}\cosh\ \alpha r\right\}, \tag{74}$$

$$F_{15}(r) = K_{15}r^5 + C_{15}\left\{\frac{\alpha^4 r^4 + 45\alpha^2 r^2 + 105}{r^5}\sinh\ \alpha r - \frac{\alpha(10\alpha^2 r^2 + 105)}{r^4}\cosh\ \alpha r\right\}, \tag{75}$$

$$F_{17}(r) = K_{17}r^7 + C_{17}\left\{\frac{\alpha^6 r^6 + 210\alpha^4 r^4 + 4725\alpha^2 r^2 + 10395}{r^7} - \frac{\alpha(21\alpha^4 r^4 + 1260\alpha^2 r^2 + 10395}{r^6}\cosh\ \alpha r\right\}. \tag{76}$$

In writing the expressions (73)–(76), the constants of integration that are multipliers in the terms of the solutions which are not defined at $r = 0$, are chosen as zero. The constants S, T, K_{14}, C_{14}, K_{15}, C_{15}, K_{17}, C_{17} are to be calculated by matching (67) with (55) at $r = 1 + 2\in\vartheta_2(\eta)$. Substituting equations (55) and (67) in the matching condition (8) and expanding F_{12}, F_{13}, f_{12}, f_{13} up to first power in $\in$ as in equation (35) and (37), we get

$$\begin{aligned}
&2F_{12}(1)\vartheta_2(\eta) + 2\in F'_{12}(1)\{\vartheta_2(\eta)\}^2 + 2F_{13}(1)\vartheta_3(\eta)\\
&+4\in F'_{13}(1)\vartheta_2(\eta)\vartheta_3(\eta) + \in F_{14}(1)\vartheta_4(\eta) + \in F_{15}\vartheta_5(\eta) + \in F_{17}(1)\vartheta_7(\eta)\\
&= 2f_{12}(1)\vartheta_2(\eta) + 4\in f'_{12}(1)\{\vartheta_2(\eta)\}^2 + 2f_{13}(1)\vartheta_3(\eta) \qquad + 4\in f'_{13}(1)\vartheta_2(\eta)\vartheta_3(\eta) + \in f_{14}(1)\vartheta_4(\eta)\\
&+4\in f'_{13}(1)\vartheta_2(\eta)\vartheta_3(\eta) + \in f_{15}(1)\vartheta_5(\eta) + \in f_{17}(1)\vartheta_7(\eta).
\end{aligned} \tag{77}$$

Substituting equation (27), $\vartheta_2\vartheta_3 = \frac{2}{3}(\vartheta_3(\eta) - \vartheta_5(\eta))$, equations (61)–(65), (68) and equations (73)–(76) in equation (77) and applying the same procedure as stated in first approximation, we get equations ($A1$), ($A2$), ($A3$), ($A4$) given in appendix. Similarly, the matching conditions (9) give the equation ($A5$), ($A6$), ($A7$), ($A8$) where M, N, S, T have been written as:

$M = M_{11} + \in M_{12}$, $N = N_{11} + \in N_{12}$, $S = S_{11} + \in S_{12}$, $T = T_{11} + \in T_{12}$.

The solutions of the components of pressure and stress are obtained for the second approximation in Re and substituting the expression for $\psi_1^{(1)}(r,\eta)$ and $\psi_1^{(2)}(r,\eta)$ from equations (55) and (67) in the matching conditions (10) and (11) at $r = 1 + 2\in\vartheta_2(\eta)$, we get twelve equations ($A7$) – ($A20$). We have calculated the values of the constants (M_{11}, N_{11}, S_{11}, T_{11}), (M_{12}, N_{12}, S_{12}, T_{12}), (A_{14}, B_{14}, C_{14}, K_{14}), (A_{15}, B_{15}, C_{15}, K_{15}), (A_{17}, B_{17}, C_{17}, K_{17}) from equations $A(1)$–$A(20)$ given in appendix for σ = 5.0, 6.0, 7.0, 8.0, 9.0, 10.0 by taking $\beta = 0.5$, $\gamma^2 = 1$ and it is represented in Table 3.

4 Results and Discussion

For motion of fluid at infinity, the stream function is given as

$$\psi_\infty = \frac{1}{2}Ur^{*2}\sin^2\theta, \tag{78}$$

The stream function $\psi - \psi_\infty$ at infinity represents a state of rest and the drag D_r^* that is employed by the fluid flow in porous oblate spheroid in our notation is given by (see, equation (4.2) of Payne and Pell (1960)):

$$D_r^* = 8\pi\mu \lim_{r^*\to\infty} \frac{r^*(c^2 U\psi^{(2)} - \psi_\infty)}{\varpi^2} \tag{79}$$

where $\bar{\omega} = r^* \sin\theta$. Substituting $\psi_0^{(2)}$ and $\psi_1^{(2)}$ from (29) and (55) respectively in the equation (23) and then in the equation (79) we get the expression for D_r^* as

$$D_r^* = 8\pi\mu c\left[(B_{01} + \in B_{02}) + \mathrm{Re}\,\frac{(B_{01} + \in B_{02})^2}{2}\right], \tag{80}$$

In case we want to write 'a' instead of c, the dimensionless drag D_r on the spheroid is given by

$$D_r = \frac{D_r^*}{6\pi\mu U a} = \frac{4}{3}\left[B_{01} + \in (B_{02} - B_{01}) + \overline{\mathrm{R}}\mathrm{e}\left\{\frac{B_{01}^2}{2} + \in B_{01}(B_{02} - B_{01})\right\}\right], \tag{81}$$

the constant c is replaced by $a(1 - \in)$ neglecting $\in^2$ and higher powers $\in$ and Re is replaced by $\overline{\mathrm{R}}\mathrm{e}$. The values of B_{01} and B_{02} when the porous spheroid is replaced by a solid one are given by assuming the velocity components to be zero at the spheroid and in this case the equations (40), (41), (43) and (44) respectively become

$$-A_{01} + B_{01} = \frac{1}{2}, \tag{82}$$

$$-A_{02} + B_{02} = -\frac{4}{5}[A_{01} + B_{01} - 1], \tag{83}$$

$$A_{01} + B_{01} = 1, \tag{84}$$

$$A_{02} + B_{02} = \frac{4}{5}[2A_{01} + 1], \tag{85}$$

The above equations give $B_{01} = \frac{3}{4}$ and $B_{02} = \frac{3}{5}$ so the dimensionless drag D_r in this case is

$$D_r = 1 - \frac{\in}{5} + \frac{3}{8}\left(1 - \frac{2}{5}\in\right)\overline{\mathrm{R}}\mathrm{e}, \tag{86}$$

In the case when $\overline{\mathrm{R}}\mathrm{e} = 0$, the expression of the drag D_r agrees with that given in the equation (4– 25.23) on page 144 by Happel and Brenner (1965) and it also agrees with that given by Payne and Pell (1960), Aoi (1955) has given the following expression for the drag coefficient for an oblate spheroid in his equation (41) as:

$$C_D = \frac{64}{RS}\left(1 + \frac{R}{25}\right), \tag{87}$$

where $S = 2\sqrt{T_0^2 + 1}\{(1 - T_0^2)\cot^{-1} T_0 + T_0\}$, $\tag{88}$

$$T_0 = \frac{b}{\sqrt{a^2 - b^2}}, \quad R = \frac{2aV}{\upsilon}. \tag{89}$$

Replacing b, R, C_D by c, 2Re, $32D_r$/Re respectively, in the equations (87)–(89) and expanding the entities in powers of $\in$ neglecting $\in^2$ and higher powers of $\in$, the equation (87) becomes exactly the equation (86). Hence our results for small Reynolds number agree completely with those of Aoi. We have calculated that the expression of drag for an oblate spheroid given by Breach (1961) when log Re is neglected becomes exactly the same as given by (81). Hence our results agree for a solid oblate spheroid with those of earlier workers, and we expect that results derived in this paper for a porous oblate spheroid are correct.

Table 1. The values of the constants (A_{01}, B_{01}, C_{01}, K_{01}), (A_{02}, B_{02}, C_{02},K_{02}), (A_{04}, B_{04}, C_{04}, K_{04}) for various values of γ for β = 0.5, – 0.5 and σ = 5

γ^2		**1.0**	**3.0**	**5.0**	**7.0**	**9.0**
β = 0.5	A_{01}	0.06073	0.1047	0.1573	0.1688	0.1704
	B_{01}	0.45765	0.5377	0.5619	0.5742	0.5763
	C_{01}	0.00028	0.0057	0.0171	0.0329	0.0524
	K_{01}	0.01831	0.0150	0.0063	0.0096	0.0117
	A_{02}	0.18890	0.3369	0.4415	0.4719	0.4810
	B_{02}	0.4782	0.5502	0.6617	0.6282	0.6326
	C_{02}	0.0011	0.0136	0.0372	0.0659	0.0985
	K_{02}	0.0346	0.0179	0.0264	0.0215	0.0181
	A_{04} B_{04}	47.6047	10.994	1.6322	0.2953	18.528
	C_{04} K_{04}	93.9205	18.648	2.0019	0.9858	3.5875
		0.00973	0.7512	0.1681	0.0075	0.0070
		71.7489	26.304	18.538	4.1573	32.428
β = –0.5	A_{01}	0.1753	0.1913	0.1981	0.2024	0.2031
	B_{01}	0.6211	0.6329	0.6387	0.6412	0.6435
	C_{01}	0.0001	0.0025	0.0081	0.0164	0.0268
	K_{01}	0.0248	0.0160	0.0112	0.0094	0.0050
	A_{02}	0.4618	0.4970	0.5134	0.5239	0.5277
	B_{02}	0.6020	0.6182	0.6272	0.6326	0.6370
	C_{02}	0.0036	0.6062	0.0165	0.0299	0.0449
	K_{02}	0.0238	0.0158	0.0143	0.0142	0.0136
	A_{04} B_{04}	1.2139	20.457	4.5609	14.946	38.225
	C_{04} K_{04}	2.4747	4.1601	12.441	33.006	78.915
		0.0005	0.2360	0.4308	1.2092	3.8276
		2.5322	7.9752	23.155	58.446	134.196

Table 2. The values of the constants (A_{01}, B_{01}, C_{01}, K_{01}), (A_{02}, B_{02}, C_{02}, K_{02}), (A_{04}, B_{04}, C_{04}, K_{04}) for various values of σ by taking γ^2 = 1 and β= 0.5, – 0.5

σ		**5.0**	**6.0**	**7.0**	**8.0**	**9.0**	**10.0**
β = 0.5	A_{01}	0.0607	0.0800	0.096140	0.109680	0.121180	0.13100100
	B_{01}	0.4580	0.5034	0.537070	0.562830	0.583150	0.59954000
	$-C_{01}$	0.0003	0.0007	0.000010	0.000003	0.000001	0.00000003
	K_{01}	0.0183	0.0140	0.010960	0.008790	0.007200	0.0059900
	A_{02}	0.1889	0.2419	0.283300	0.317800	0.346000	0.3693000
	B_{02}	0.4782	0.5083	0.536100	0.545800	0.553400	0.5602000
	C_{02}	0.0110	0.0003	0.000050	0.000018	0.000006	0.0000020
	K_{02}	0.0346	0.0236	0.007600	0.009300	0.010100	0.0094000
	$-A_{04}$	47.605	17.986	3912.940	273.4835	568.0042	873.20990
	B_{04}	93.921	28.932	6150.740	346.9285	772.2964	1368.0607
	C_{04}	0.0097	0.0004	0.011230	0.000098	0.000055	0.0000191
	K_{04}	71.749	16.369	3093.010	136.6345	264.5873	417.62170

σ		5.0	6.0	7.0	8.0	9.0	10.0
β = –0.5	A_{01}	0.1767	0.18471	0.187930	0.198120	0.020253	0.2068100
	B_{01}	0.6211	0.64514	0.660140	0.674490	0.683520	0.6911700
	$-C_{01}$	0.0001	0.00002	0.000004	0.000001	0.000003	0.0000001
	K_{01}	0.0248	0.01792	0.011354	0.010530	0.008440	0.0069000
	A_{02}	0.4618	0.48159	0.488780	0.507990	0.516950	0.5250800
	B_{02}	0.6018	0.602142	0.602630	0.602810	0.601820	0.6011500
	C_{02}	0.0004	0.000095	0.000022	0.000006	0.000002	0.0000007
	$-K_{02}$	0.0238	0.017064	0.013540	0.011610	0.008400	0.0062800
	$-A_{04}$	1.2139	–871.5602	–9527.44	1317.461	1864.026	20379.584
	B_{04}	2.4747	1538.3674	16679.77	2628.862	3462.252	4264.6972
	C_{04}	0.0005	0.003417	0.055810	0.002342	0.000895	0.0001876
	K_{04}	2.5322	1183.6944	19745.65	2122.700	2968.0067	3664.0963

Table 3. The values obtained for the constants in the solution of the second approximation by taking γ^2 = 1.0 and β = 0.5 for various values of σ

σ	5.0	6.0	7.0	8.0	9.0	10.0
$-S_{11}$	0.02434	0.018693	0.007163	0.003926	0.0009578	0.0004289
T_{11}	0.000023	0.000005	0.00000072	0.000000145	0.000000073	0.0000000538
$-M_{11}$	0.06158	0.07233	0.14697	0.18543	0.28345	0.42894
N_{11}	0.055353	0.0523	0.116255	0.13725	0.24009	0.38935
$-S_{12}$	6.2953	2.41776	0.98524	4.7233	16.5853	84.7823
T_{12}	0.008392	0.0012	0.009457	0.0006289	0.006833	0.0004478
$-M_{12}$	2.32594	32.3323	73.8558	332.0043	568.5968	401.0989
N_{12}	4.6374	20.81466	63.1957	173.4453	351.0607	268.5867
A_{14}	53.8954	167.9234	478.5432	875.0089	2437.893	56675875
$-B_{14}$	478.0985	194.7376	136.5863	97.5387	44.8875	12.8937
C_{14}	0.0008934	0.0004137	0.0000985	0.00005863	0.000029	0.00000795
$-K_{14}$	12.7269	44.7595	184.4462	568.9432	863.4458	2263.4458
A_{15}	1262.8458	3582.934	8456.2248	23378.009	48564.643	89578.0908
$-B_{15}$	8503.1004	3102.1289	2278.1142	873.589	684.7895	124.8975
C_{15}	0.09859	0.0032107	0.02958	0.008464	0.002375	0.0007354
$-K_{15}$	378.9452	1290.9943	3678.0912	5855.6600	9585.1189	24385.0092
$-A_{17}$	685.0092	472.958	368.9463	273.1890	184.5672	94.0667
B_{17}	384.4462	429.2049	823.5032	1765.7758	3875.8463	9575.8968
C_{17}	0.000086	0.000005	0.0000012	0.00000094	0.0000007	0.00000023
K_{17}	12.5865	80.99245	348.6675	1585.49	4485.6853	12375.9960

Table 4. The values of D_r for different values of ϵ and Re by taking β = 0.50, σ = 10.0 and γ^2 = 3.0

γ^2	1.0					3.0				
Re / ϵ	1.0	1.5	2.0	2.5	3.0	1.0	1.5	2.0	2.5	3.0
0.0	1.03901	1.15883	1.27865	1.3985	1.5128	1.1689	1.3138	1.4587	1.6036	1.7486
0.04	1.03566	1.15485	1.27403	1.39322	1.5124	1.1642	1.3082	1.4522	1.5962	1.7407
0.08	1.03230	1.15086	1.26942	1.38798	1.5065	1.1595	1.3026	1.4456	1.5887	1.7317
0.12	1.02895	1.14688	1.26480	1.38273	1.5006	1.1548	1.2970	1.4391	1.5811	1.7233
0.16	1.02560	1.14289	1.26019	1.37750	1.4948	1.1502	1.2913	1.4325	1.5740	1.7150
0.20	1.02224	1.13890	1.25560	1.37720	1.4889	1.1455	1.2857	1.4260	1.5662	1.7065

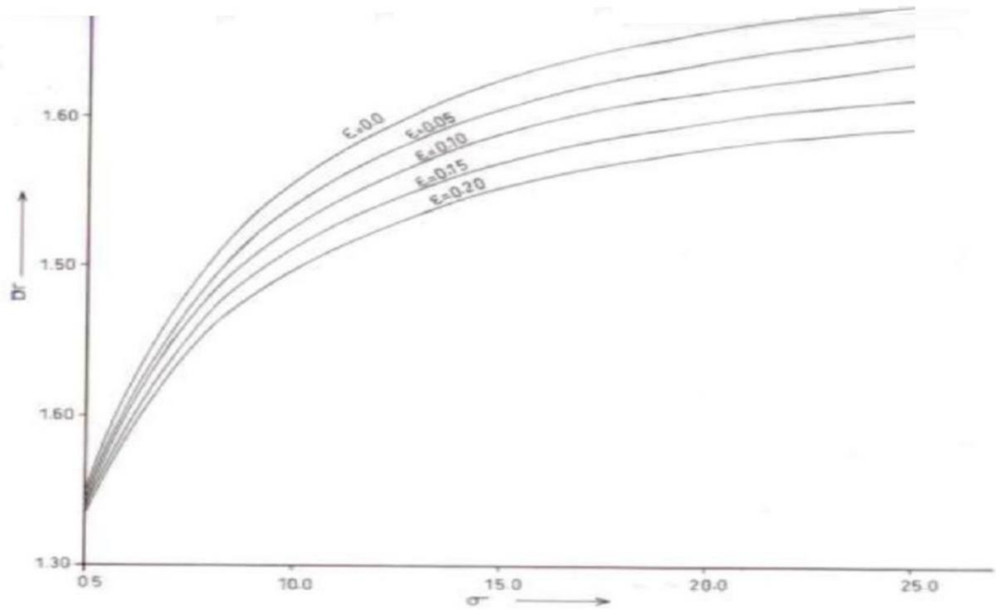

Fig. 2. The graph of Dr against σ taking $\gamma^2 = 1$ and β = 0.5

A graph of drag D_r against σ is drawn in Fig. 2 for various values of $\in$ by taking $\gamma^2 = 1$ and $\beta = -0.5$. It reveals that drag reduces with lead of $\in$ i.e. with the departure of the shape of spheroid from that of a sphere. The drag on the oblate porous spheroid due to moving fluid increases with the lead of σ, i.e. it reduces with the lead of parameter of permeability of porous medium. The graph of the drag D_r against the departure of the shape of the spheroid $\in$ for different values of Reynolds number has been plotted in Figs. 3 & 4 which show that the drag on the spheroid increases with the increase of large as well as small values of Reynolds number and decreases with the increase of $\in$. The values of the drag D_r for various values of parameter of departure of the shape of spheroid $\in$ and Re have been given in Table 4 for $\gamma^2 = 1.0, 3.0$, $\beta = 0.5$ and $\sigma = 10.0$. This table represents that the drag leads with the lead of Re and γ^2 but decreases with the lead of $\in$. Though the formula (86) for the drag is presume valid for little rates of $\in$ but in fact it is supposed to be exact for even huge withdrawal from the sphere structure (see, Happel and Brenner 1965).

5 Conclusions

The main objective is to study of the flow of viscous fluid past a porous oblate spheroid at a small Reynolds number, and for this purpose the perturbation technique is used. Here the graph of drag D_r against departure of shape of spheroid $\in$ for different values of Reynolds number has been plotted in Figs. 3 & 4 which represent the drag on the spheroid leads with the lead of small values of Reynolds number and lacks with the leads of $\in$, which is given in Fig. 3 and Fig. 4. In the Fig. 2 which represents the drag reduces with rise of the $\in$ i.e. with the change of the shape of the spheroid from that of a sphere. Here the drag on the oblate porous spheroid due to flow of fluids leads with the lead of σ, i.e. it lacks with the leads of the permeability parameter of the porous materials. The application of a viscous fluid flowing past a porous oblate spheroid at a small Reynolds number is to solve the problem of uniform steady viscous flow over an oblate in the low range of Reynolds number. Some applications of the Reynolds number included the calculation of friction factor and drag in internal and external flow, hydraulic study, aero foil design, filtration technology, geothermal energy, and precipitation.

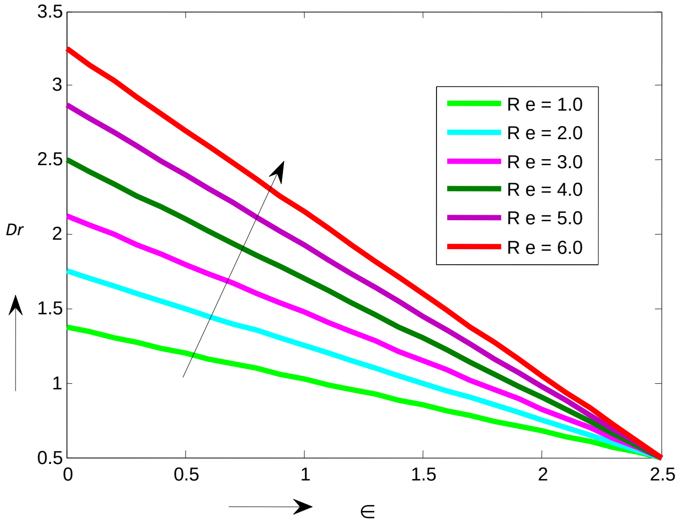

Fig. 3. The graph between *Dr* against ∈ and the variation of Reynolds Number

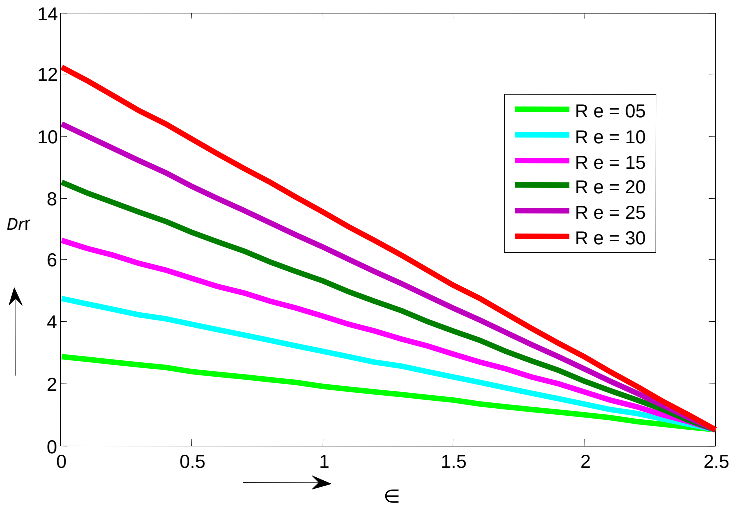

Fig. 4. The graph between Dr against ∈ and the variation of Reynolds Number

Disclaimer (Artificial Intelligence)

Authors (Dr. Pravin Kumar Srivastava, Dr. Ramesh Yadav, Mr. Ravindra Pratap Pandey and Dr. Nidhi Pandya) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

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Competing Interests

Authors have declared that no competing interests exist.

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Appendix

$$S_{11} + T_{11}\{(3+\alpha^2)\sinh\alpha - 3\alpha\cosh\alpha\} = M_{11} + N_{11} + \frac{B_{01}}{4}\left[2A_{01} + 2B_{01} - 1\right], \tag{A1}$$

$$\frac{4}{7}[3S_{11} - T_{11}\{(3\alpha^2+16)\sinh\alpha - (\alpha^3+16\alpha)\cosh\alpha\}] + [S_{12} + T_{12}\{(3+\alpha^2)\sinh\alpha - 3\alpha\cosh\alpha\}]$$
$$= \frac{4}{7}\left[-2M_{11} + \frac{B_{01}}{2}(-2A_{01} + B_{01} - 1)\right] + M_{12} + N_{12} + B_{04}\left[\frac{425}{21168}B_{04} + \frac{5}{1008}A_{04}\right], \tag{A2}$$

$$-\frac{4B_{01}}{5}[2K_{01} - C_{01}\{(\alpha^2+1)\sinh\alpha - \alpha\cosh\alpha\}]$$
$$+[K_{14} + C_{14}\{(6\alpha^2+15)\sinh\alpha - \alpha(\alpha^2+15)\cosh\alpha\}]$$
$$= -\frac{2B_{01}}{5}(-A_{01} - B_{01} + 1) + A_{14} + B_{14} - B_{04}\left\{\frac{10}{231}B_{04} + \frac{25}{1089}A_{04}\right\}, \tag{A3}$$

$$-\frac{8}{7}[3S_{11} - T_{11}\{(3\alpha^2+16)\sinh\alpha - (\alpha^3+6\alpha)\cosh\alpha\}]$$
$$+[K_{15} + C_{15}\{(\alpha^4+45\alpha^2+105)\sinh\alpha - \alpha(10\alpha^2+105)\cosh\alpha\}]$$
$$= -\frac{8}{7}\left[-2M_{11} + \frac{B_{01}}{2}(-A_{01} + B_{01} - 1)\right] + A_{15} + B_{15} + B_{04}\left\{-\frac{15}{56}B_{04} + \frac{25}{252}A_{04}\right\}, \tag{A4}$$

$$K_{17} + C_{17}\{(\alpha^6 + 210\alpha^4 + 472\alpha^2 + 10395)\sinh\alpha - \alpha(21\alpha^4 - 1260\alpha^2 + 10395)\cosh\alpha\}$$
$$= A_{17} + B_{17} + B_{04}\left\{\frac{125}{2376}B_{04} - \frac{125}{792}A_{04}\right\}, \tag{A5}$$

$$3S_{11} - T_{11}\{(6+3\alpha^2)\sinh\alpha - (6\alpha+\alpha^3)\cosh\alpha\} = -2M_{11} + \frac{B_{01}}{4}(-A_{01} + B_{01} - 1), \tag{A6}$$

$$\frac{4}{7}[6S_{11} + T_{11}\{(\alpha^4+9\alpha^2+18)\sinh\alpha - (3\alpha^3+18\alpha)\cosh\alpha\}]$$
$$+[3S_{12} - T_{12}\{(6+3\alpha^2)\sinh\alpha - (6\alpha+\alpha^3)\cosh\alpha\}] = \frac{4}{7}\left[6M_{11} + \frac{B_{01}}{2}(2A_{01} - 1)\right],$$
$$-\left[2M_{12} + B_{04}\left\{\frac{1315}{21168}B_{04} + \frac{25}{1008}A_{04}\right\}\right] \tag{A7}$$

$$-\frac{4B_{01}}{5}[2K_{01} + C_{01}\{(\alpha^2+2)\sinh\alpha - (\alpha^3+2\alpha)\cosh\alpha\}]$$
$$+[4K_{14} - C_{14}\{(\alpha^4+21\alpha^2+45)\sinh\alpha - (6\alpha^3+45\alpha)\cosh\alpha\}]$$
$$= -\frac{4B_{01}}{5}(2A_{01}+1) - (3A_{14} + B_{14}) + B_{04}\left\{\frac{40}{251}B_{04} + \frac{125}{1089}A_{04}\right\}, \tag{A8}$$

$$
\begin{aligned}
&-\frac{8}{7}[6S_{11}+T_{11}\{(\alpha^4+9\alpha^2+18)\sinh\alpha-(3\alpha^3+18\alpha)\cosh\alpha\}] \\
&+[5K_{15}-C_{15}\{11\alpha^4+240\alpha^2+525\}\sinh\alpha-(\alpha^5+65\alpha^3+525\alpha)\cosh\alpha\}] \\
&=-\frac{8}{7}\left[6M_{11}+\frac{B_{01}}{2}(2A_{01}+1)\right]-(A_{15}+2B_{15})+B_{04}\left\{\frac{45}{56}B_{04}-\frac{125}{252}A_{04}\right\},
\end{aligned}
\tag{A9}
$$

$$
\begin{aligned}
&-7K_{17}-C_8\{(22\alpha^6+1890\alpha^4+34020\alpha^2+72765)\sinh\alpha \\
&-(\alpha^7+252\alpha^5+9765\alpha^3+72765\alpha)\cosh\alpha\} \\
&=-(6A_{17}+4B_{17})+B_{04}\left\{-\frac{125}{792}B_{04}+\frac{625}{792}A_{04}\right\},
\end{aligned}
\tag{A10}
$$

$$
\begin{aligned}
&\gamma^2[6S_{11}+T_{11}\{(\alpha^4+21\alpha^2+48)\sinh\alpha-(5\alpha^3+48\alpha)\cosh\alpha\}]-16M_{11}-6N_{11}-B_{01}(10A_{01}+4B_{01}) \\
&=\beta\sigma\left[-2M_{11}+\frac{B_{01}}{2}(A_{01}+B_{01}-1)\right],
\end{aligned}
\tag{A11}
$$

$$
\begin{aligned}
&\frac{4}{7}\gamma^2[6S_{11}-T_{11}\{90\alpha^2+5\alpha^4+192)\sinh\alpha-(\alpha^5+26\alpha^3+192\alpha)\cosh\alpha\}] \\
&+\gamma^2[6S_{12}+T_{12}\{(\alpha^4+21\alpha^2+48)\sinh\alpha-(5\alpha^3+48\alpha)\cosh\alpha\}] \\
&+\frac{4}{7}[64M_{11}+12N_{11}+B_{01}(30A_{01}+4B_{01})-\left[16M_{12}+6N_{12}+B_{04}\left(\frac{3805}{5292}B_{04}+\frac{115}{504}A_{04}\right)\right] \\
&=\beta\sigma\left[\frac{4}{7}\left\{6M_{11}+\frac{B_{01}}{2}(2A_{01}-1)\right\}-2M_{12}-B_{04}\left\{\frac{1315}{21168}B_{04}+\frac{25}{1008}A_{04}\right\}\right],
\end{aligned}
\tag{A12}
$$

$$
\begin{aligned}
&\frac{8}{5}\gamma^2\,\frac{B_{01}C_{01}}{2}\{(\alpha^4+9\alpha^2+18)\sinh\alpha-(3\alpha^3+18\alpha)\cosh\alpha\} \\
&+\gamma^2[16K_{14}+C_{14}\{(8\alpha^4+210\alpha^2+450)\sinh\alpha-(\alpha^5+51\alpha^3+450\alpha)\cosh\alpha\}] \\
&-\frac{72}{5}B_{01}A_{01}-(30A_{14}+16B_{14})+B_4\left\{\frac{400}{231}B_4+\frac{1300}{1089}A_4\right\} \\
&\beta\sigma\left[-\frac{2}{5}B_{01}(2A_{01}+1)-3(A_{14}+B_{14})+B_{04}\left\{\frac{40}{231}B_{04}+\frac{125}{1089}A_{04}\right\}\right],
\end{aligned}
\tag{A13}
$$

$$
\begin{aligned}
&-\frac{8}{7}\gamma^2[6S_{11}-T_{11}\{(5\alpha^4+90\alpha^2+192)\sinh\alpha-(\alpha^5+26\alpha^3+192\alpha)\cosh\alpha\}] \\
&+\gamma^2[30K_{15}+C_{15}\{(\alpha^6+129\alpha^4+2865\alpha^2+6300)\sinh\alpha \\
&-(14\alpha^5+765\alpha^3+6300\alpha)\cosh\alpha]-\frac{8}{7}\{64M_{11}+12N_{11}+B_{01}(30A_{01}+4B_{01})\} \\
&-\left[48A_{15}+30B_{15}+B_{04}\left\{-\frac{165}{7}B_{04}+\frac{325}{63}A_{04}\right\}\right] \\
&=\beta\sigma\left[-\frac{8}{7}\left\{6M_{11}+\frac{B_{01}}{2}(2A_{01}-1)\right\}-4A_{15}-2B_{15}+B_{04}\left\{\frac{45}{56}B_{04}-\frac{125}{252}A_{04}\right\}\right],
\end{aligned}
\tag{A14}
$$

$$\gamma^2[70K_{17} + C_{17}\{(\alpha^8 + 382\alpha^6 + 29925\alpha^4 + 294178\alpha^2 + 1164240)\sinh\alpha$$
$$-(25\alpha^7 + 4032\alpha^5 + 155295\alpha^3 + 1164240\alpha)\cosh\alpha]$$
$$-\left[96A_{17} + 74B_{17} + B_{04}\left\{\frac{625}{198}B_{04} - \frac{5125}{396}A_{04}\right\}\right] = \beta\sigma\left[-(6A_{17} + 4B_{17}) + B_{04}\left\{\frac{25}{792}B_{04} + \frac{625}{792}A_{04}\right\}\right], \tag{A15}$$

$$\frac{\gamma^2}{6}[12S_{11} + T_{11}\{(3\alpha^4 - 54\alpha^2 - 144)\sinh\alpha - (\alpha^5 - 6\alpha^3 - 144\alpha)\cosh\alpha\}]$$
$$+\frac{\sigma^2}{6}[3S_{11} - T_{11}\{(6 + 3\alpha^2)\sinh\alpha - (6\alpha + \alpha^3)\cosh\alpha\}]$$
$$= -2(N_{11} + A_{01}B_{01}) + 2\left[-4M_{11} - 2N_{11} - \frac{B_{01}}{2}(3A_{01} + B_{01})\right], \tag{A16}$$

$$\frac{\gamma^2 T_{11}}{42}\{(\alpha^6 + 49\alpha^4 + 1098\alpha^2 + 2400)\sinh\alpha - (3\alpha^5 + 298\alpha^3 + 2400\alpha)\cosh\alpha\}$$
$$+\frac{\gamma^2}{6}[12S_{12} - T_{12}\{(3\alpha^4 - 54\alpha^2 - 144)\sinh\alpha - (\alpha^5 - 6\alpha^3 - 144\alpha)\cosh\alpha\}]$$
$$+\frac{\sigma^2}{42}[6S_{11} + T_{11}\{(\alpha^4 + 9\alpha^2 + 18)\sinh\alpha - (3\alpha^3 + 18\alpha)\cosh\alpha\}]$$
$$+\frac{\sigma^2}{6}[3S_{12} - T_{12}\{(6 + 3\alpha^2)\sinh\alpha - (6\alpha + \alpha^3)\cosh\alpha\}] = -\frac{6}{21}(3N_{11} + 4A_{01}B_{01}) - 2(N_{12} + A_{01}B_{01})$$
$$+\frac{20}{21}\{20M_{11} + 6N_{11} + B_{01}(12A_{01} + 2B_{01})\} - 2\left[4M_{12} + 2N_{12} + B_{04}\left\{\frac{2165}{21168}B_{04} + \frac{5}{144}A_{04}\right\}\right], \tag{A17}$$

$$\gamma^2\frac{B_{01}C_{01}}{3}\{(\alpha^4 - 18\alpha^2 - 48)\sinh\alpha - (\alpha^5 - 2\alpha^3 - 48\alpha)\cosh\alpha\}$$
$$-\frac{\sigma^2}{3}B_{01}[2K_{01} + C_{01}\{(\alpha^2 + 2)\sinh\alpha - (\alpha^3 + 2\alpha)\cosh\alpha\}]$$
$$+\frac{\sigma^2}{12}[4K_{14} - C_{14}\{(\alpha^4 + 21\alpha^2 + 45)\sinh\alpha - (6\alpha^3 + 45\alpha)\cosh\alpha\}]$$
$$+\frac{\gamma^2}{2}[4K_{14} + C_{14}\{(\alpha^6 + 17\alpha^4 - 87\alpha^2 - 300)\sinh\alpha - (6\alpha^5 + 13\alpha^3 - 300\alpha)\cosh\alpha\}]$$
$$= \frac{4}{3}B_{01}^2 - \frac{1}{12}\left[30B_{14} + B_{04}\left\{\frac{166}{177}B_{04} + \frac{350}{121}A_{04}\right\}\right]$$
$$-\frac{4}{3}B_{01}(12A_{01} - 2B_{01}) + 2\left[-5A_{14} - 3B_{14} + B_{04}\left\{\frac{20}{77}B_{04} + \frac{175}{1089}A_{04}\right\}\right], \tag{A18}$$

$$\frac{\gamma^2}{70}T_{11}\{(\alpha^6-3\alpha^4-306\alpha^2-720)\sinh\alpha-(3\alpha^5-66\alpha^3-720\alpha)\cosh\alpha\}$$
$$+\frac{\gamma^2 C_{15}}{160}\{(13\alpha^6+436\alpha^4+3975\alpha^2+7350)\sinh\alpha-(\alpha^7+91\alpha^5+1525\alpha^3+7350\alpha)\cosh\alpha\}$$
$$-\frac{\sigma^2}{70}[6S_{11}+T_{11}\{(\alpha^4+9\alpha^2+18)\sinh\alpha-(3\alpha^3+18\alpha)\cosh\alpha\}]$$
$$+\frac{\sigma^2}{160}[5K_{15}-C_{15}\{(11\alpha^4+240\alpha^2+525)\sinh\alpha-(\alpha^5+65\alpha^3+525\alpha)\cosh\alpha\}]$$
$$+\frac{\gamma^2}{4}[3K_{15}-C_{15}\{(13\alpha^4+330\alpha^2+735)\sinh\alpha-(\alpha^5+85\alpha^3+735\alpha)\cosh\alpha\}]$$
$$=-\frac{6}{35}(3N_{11}+4A_{01}B_{01})-\frac{1}{160}\left[56B_{15}B_{04}\left\{\frac{151}{14}B_{04}+\frac{125}{18}A_{04}\right\}\right]$$
$$-\frac{6}{35}[20M_{11}+6N_{11}+B_{01}(12A_{01}+2B_{01})]+\frac{1}{4}\left[-6A_{15}-4B_{15}+B_{04}\left\{\frac{75}{56}B_{04}-\frac{25}{36}A_{04}\right\}\right], \tag{A19}$$

$$\frac{\gamma^2}{672}[420K_{17}+C_{17}\{(24\alpha^8+510\alpha^6-121590\alpha^4-3251934\alpha^2$$
$$-6548850)\sinh\alpha-(\alpha^9+216\alpha^7+9135\alpha^5-773955\alpha^3-6548850\alpha)\cosh\alpha\}]+\frac{\sigma^2}{672}[7K_{17}$$
$$-C_{17}\{22\alpha^6+1890\alpha^4+34020\alpha^2+72765)\sinh\alpha-(\alpha^7+252\alpha^5+9765\alpha^3+72765\alpha)\cosh\alpha]$$
$$=-\frac{1}{672}\left[108B_{17}+B_{04}\left\{\frac{3125}{396}B_{04}-\frac{875}{66}A_{04}\right\}\right]-\frac{1}{8}\left[8A_{17}+6B_{17}+B_{04}\left(\frac{625}{2376}\right)-\frac{875}{792}A_{04}\right], \tag{A20}$$

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