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# Fixed Point Results for Contractive Mappings in $F$-metric Space

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***Authors' contributions***

*This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.*

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## Abstract

In this research, the existence of fixed point and it′s uniqueness were investigated in the setting of F-metric spaces, taking into account certain contractive conditions such as p-contraction and monotonically decreasing contraction. The findings extend and generalize a number of previously published findings.

*Keywords:* F−*metric space*; *contraction*; *existence and uniqueness*; F−*Cauchy*.

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# 1 Introduction

One of the more conventional theories in functional and nonlinear analysis is fixed point theory. Numerous mathematicians have expanded the metric structure by altering Frechet's initial notion of a metric after he created metric spaces (Fréchet, 1906). The triangle inequality of the original formulation is modified in order to make the majority of the generalizations.

In the 1960s, (Ghaler, 1963) defined 2-metric space in a series of publications, which he claimed was a generalization of ordinary metric spaces. This is one of the well-known metrics of such generalizations. This space's structure is described as follows:

Let $Y$ be a nonempty set. A function $b : Y \times Y \to \mathbb{R}$ is said to be a 2-metric on $Y$ if it satisfies the following properties:

1 For distinct points $i, j \in Y$, there is a point $K \in Y$ such that $H(i, j, k) \neq 0$,

2 $b(i, j, k) = 0$ if any two elements of the triplet $(i, j, k)$ are equal,

3 $(i, j, k)$=$(i, j, k)$....(symmetry),

4 $(i, j, k) \leq (i, j, a) + (i, a, k) + (a, j, k)$ for all $i, j, k, a \in Y$ (triangle inequality).

A generalized metric space and associated fixed point theorems were defined by (Jleli and Samet, 2015). More recently, (Jleli and Samet, 2018) presented another abstraction of this type, which they call $F$-metric.

Let $f \in F$ and $f : (0, +\infty) \to \mathbb{R}$ be such that:
$(F1)$ $0 < x < y \implies f(x) \leq f(j)$; and
$(F2)$ for $\{i_n\} \subseteq R^+$, $\lim_{n\to\infty} i_n = 0 \leftrightarrow \lim_{n\to\infty} f\{i_n\} = -\infty$

**Definition 1** Let $d_F : M \times M \to [0, +\infty)$ be a given mapping, and let $M$ be a nonempty set. Assume that $(f, \alpha) \in F \times [0, +\infty)$ occurs in such a way that:

1. $H(i, j) \in M \times M, H(i, j) = 0 \implies x = y$;
2. $H(i, j) = H(i, j)$ for all $(x, y) \in M \times M$; and
3. for every $(x, y) \in M \times M, N \in \mathbb{N}, N \geq 2$, and $(i_i)_i^n = 1 \subset M$ with $(i_1, i_N) = (x, y)$, we get

$$H(i, j) > 0 \ \ implies \ \ f(H(i, j)) \leq f(\sum_{i=1}^{N-1} H(i_i, y_{i+1})) + \alpha$$

The pair $(M, H)$ is then referred to as a $F$-metric space, and $H$ is a $F$-metric on $M$.

The approach of successive approximations used to prove the existence of solutions of differential equations, which was independently introduced by (Liouville, 1837) and (Picard, 1890), is where fixed point theory got its start. However, it was formally introduced as a significant component of analysis at the turn of the 20th century. The famous Polish mathematician's groundbreaking work is the abstraction of this classical theory.

The concept of sequential $F$-metric spaces was introduced as an extension of normal metric spaces, b-metric spaces, JS-metric spaces, and mainly $F$-metric spaces (Roy et al., 2021) . We looked at some of these spaces' topological properties. Considering this notion, they proved fixed-point theorems for some classes of contractive mappings over such spaces. Their fixed-point theorems are supported by examples, which also verify that the underlying space is valid. Furthermore, their fixed-point theorem is used to solve a system of linear algebraic equations.

Sharma and Chandok (Sharma and Chandok, 2022), presented a new class of Picard operators for such mappings in the framework of $F$ -metric space and examined a fixed point problem associated with specific contraction mappings, producing some interesting and original findings. They also showed that the integral equation and fixed point issue are well-posed, and they examined the Hyers-Ulam stability of an integral equation, a Cauchy functional equation, and a fixed point problem as applications of their findings (see. (Alansari et al., 2020), (Hussain and Khan, 2020), (Alnaser et al., 2019), (Ashis et al., 2019)).

# 2 Preliminary

This section will examine definitions, examples, lemmas, propositions, and properties that are crucial in producing our main findings.

Jleli and Samet (Jleli and Samet, 2018) coined the concept of $F$-metric spaces by utilizing a particular class of auxiliary functions, which we start with.

Let $f \in F$ and $f : (0, +\infty) \to \mathbb{R}$ be such that:
$(F1)$ $0 < i < j \implies f(i) \leq f(j)$; and
$(F2)$ for $\{i_n\} \subseteq R^+$, $\lim\limits_{n\to\infty} i_n = 0 \leftrightarrow \lim\limits_{n\to\infty} f\{i_n\} = -\infty$

**Example 2.1** (Jleli and Samet, 2018) Examples of the type of auxiliary functions that were previously addressed include the following:

i $-\frac{1}{t}$ where $t \in (0, \infty)$;

ii $-\exp^{\frac{1}{t}}$ for all $t \in (0, \infty)$.

The authors developed the idea of $F$ -metric spaces by using such functions to broaden the idea of conventional metric spaces (Jleli and Samet, 2018).

**Definition 2.1** Let $d_F : M \times M \to [0, +\infty)$ be a given mapping, and let $M$ be a nonempty set. Assume that $(f, \alpha) \in F \times [0, +\infty)$ exists in such a way that:

H1 $H(i, j) \in M \times M, H(i, j) = 0 \implies x = y$;

H2 $H(i, j) = H(i, j)$ for all $H(i, j) \in M \times M$; and

H3 for every $(i, j) \in M \times M, N \in \mathbb{N}, N \geq 2$, and $(i_i)_i^n = 1 \subset M$ with $(i_1, i_N) = (i, j)$, we get

$$H(i,j) > 0 \ \ implies \ \ f(H(i,j)) \leq f(\sum_{i=1}^{N-1} H(i_n, y_{n+1})) + \alpha$$

Then $H$ is an $F$-metric on $M$, and the pair $(M, H)$ is said to be an $F$-metric space.

It is observed that any metric on $Y$ is an $F$ -metric, but the converse is not true.

**Proposition 2.1**.(Jleli and Samet, 2018)

Let $(j, H)$ be a space with $F$ metrics. Let $i \in Y$ and $\{i_n\}$ be a sequence in $Y$. The statements that follow are interchangeable:

i $\{i_n\}$ is $F$-convergent to $i$,

ii $\lim\limits_{i_n\to\infty} H(i_n, i) = 0$

The next result shows that the limit of an F-convergent sequence is unique.

**Proposition 2.2** (Jleli and Samet, 2018)
Let $(j, H)$ be an $F$-metric space. Let $\{i_n\}$ be a sequence in $Y$.

Then

$$(i, j) \in Y \times Y, \lim_{i_n \to \infty} H(i_n, i) = \lim_{i_n \to \infty} H(i_n, j) = 0 \Rightarrow i = j$$

**Definition 2.2** (Jleli and Samet, 2018)

Let $(j, H)$ be an $F$-metric space. Let $\{i_n\}$ be a sequence in $Y$.

i We say that $\{i_n\}$ is $F$-convergent, if
$$\lim_{i_n \to \infty} H(i_n, x) = 0$$
ii We say that $i_n$ is F-Cauchy, if
$$\lim_{i_n \to \infty} H(i_n, i_m) = 0$$
iii We say that $(j, H)$ is $F$-complete, if every $F$-Cauchy sequence in $Y$ is $F$- convergent to a certain element in $Y$.

**Proposition 2.3** (Jleli and Samet, 2018)

Let $(j, H)$ be an $F$-metric space. If $\{i_n\} \subset Y$ is $F$-convergent, then it is $F$-Cauchy.

# 3 Main Results

We state our main results as follows.

**Theorem 3.1**

Let $(j, H)$ be an $F$-metric space and $g : Y \to Y$ be a continuous $p$-contraction mapping. Then $g$ has a unique fixed point in $Y$

$$H(gi, gj) \leq k[H(i, j) + |H(i, gi) - H(j, gj)|] \tag{1}$$

***Proof***

Let $i_0 \in Y$ be an arbitrary point and define a sequence $\{i_n\}$ by $i_{n+1} = gi_n$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $i_{n_0} = i_{n_0+1}$ is a fixed point of $g$. Suppose $i_n \neq i_{n+1}$ for all $n \in \mathbb{N}$ hence $H(i_n, i_{n+1}) > 0$

Now we have.

$$\begin{align} H(i_{n+1}, i_{n+2}) &= H(gi_n, gi_{n+1}) \tag{2}\\ &\leq k[H(i_n, i_{n+1}) + |H(i_n, gi_n) - H(i_{n+1}, gi_{n+1})|] \tag{3}\\ &= k[H(i_n, i_{n+1}) + |H(i_n, i_{n+1}) - H(i_{n+1}, i_{n+2})|] \tag{4}\end{align}$$

for all $n \in \mathbb{N}$ if $H(i_n, i_{n+1}) \geq H(i_{n+1}, i_{n+2})$ for some $n$ then from (1)

$$\begin{align} H(i_{n+1}, i_{n+2}) &\leq k[H(i_n, i_{n+1}) + |H(i_n, i_{n+1}) - H(i_{n+1}, i_{n+2})|] \tag{5}\\ &\leq k[H(i_n, i_{n+1}) + H(i_n, i_{n+1}) - H(i_{n+1}, i_{n+2})] \tag{6}\\ &= 2kH(i_n, i_{n+1}) - kH(i_{n+1}, i_{n+2}) \tag{7}\end{align}$$

$$H(i_{n+1}, i_{n+2}) + kH(i_{n+1}, i_{n+2}) \leq 2kH(i_n, i_{n+1}) \tag{8}$$
$$(1+k)H(i_{n+1}, i_{n+2}) \leq 2kH(i_n, i_{n+1}) \tag{9}$$
$$H(i_{n+1}, i_{n+2}) \leq \frac{2k}{1+k}H(i_n, i_{n+1}) \tag{10}$$

for all $n \in \mathbb{N}$

Let $\frac{2k}{1+k} = \lambda$, then $0 < \lambda < 1$ and so we have

$$\begin{aligned} H(i_{n+1}, i_{n+2}) &\leq \lambda H(i_n, i_{n+1}) && (11)\\ &\leq \lambda^2 H(i_{n-1}, i_n) && (12)\\ &\leq \lambda^3 H(i_{n-3}, i_{n-1}) && (13)\\ &. && (14)\\ &. && (15)\\ &. && (16)\\ &\leq \lambda^n H(i_1, i_0) && (17) \end{aligned}$$

for all value of $n \in \mathbb{N}$

We have for $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} \sum_{i=n}^{m-1} H(i_n, i_{n+1}) &\leq \sum_{i=n}^{m-1} \lambda^n H(i_0, i_1) && (18)\\ &\leq \frac{\lambda^n}{1-\lambda} H(i_0, i_1) && (19) \end{aligned}$$

since $0 < \lambda < 1$ for all $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > no_0$

$$0 < \frac{\lambda^n}{1-\lambda} H(i_0, i_1) < \delta \tag{20}$$

Now, let $(f, \alpha) \in F \times [0, \infty)$ be such that (H3) is satisfied.

let $\epsilon > 0$ be fixed, then by (f2) there exists $\eta > 0$ such that

$$0 < t < \eta \Rightarrow f(t) < f(\epsilon) - \alpha \tag{21}$$

considering $\delta$ as $\eta$ we get

$$f\left(\frac{\lambda^n}{1-\lambda} H(i_0, i_1)\right) < f(\epsilon) - \alpha \tag{22}$$

By (f1) we have

$$f\left(\sum_{i=1}^{N-1} H(i_i, i_{i+1})\right) \leq f\left(\frac{\lambda^n}{1-\lambda} H(i_0, i_1)\right) < f(\epsilon) - \alpha \tag{23}$$

for all $m, n \in \mathbb{N}$ with $m > n \geq n_0$.

Using (H3) and equation (23) we get for $m > n \geq n_0$

$$H(i_n, i_m) > 0 \Rightarrow f\left(H(i_n, i_m)\right) \leq f\left(\sum_{i=1}^{m-1} H(i_i, i_{i+1})\right) + \alpha < f(\epsilon) \tag{24}$$

which implies by (f1) that $H(i_n, i_m) < \epsilon$ for $m > n \geq n_0$. Therefore the sequence $\{i_n\}$ is $F$-Cauchy.

Since $H(i,j)$ is $F$-complete there exists $k \in Y$ such that $\{i_n\}$ is $F$-convergent to $z$, that is $\lim_{n\to\infty} H(i_{n+1}, z) = 0$

Since $g$ is continuous, and the $\lim_{n\to\infty}(gi_n, gk) = \lim_{n\to\infty} H(i_{n+1}, gk) = 0$

The uniqueness of the limit we have $z = gk$.

Now suppose $w$ is another fixed point of $g$, then $gw = w$ and $H(k, w) > 0$.

Hence

$$\begin{aligned} H(k,w) &= H(gk, gw) &(25)\\ &\leq k[H(k,w) + |H(k,gk) - H(w,gw)|] &(26)\\ &= kH(k,w) &(27) \end{aligned}$$

which is a contradiction, hence the fixed point of g is unique.

**Theorem 3.2**

Let $g$ be a self mapping on an $F$-metric space $H(j, H)$. Suppose there exists $i_0 \in Y$ such that

$$\phi(H(gi, gj)) < \frac{1}{2}\{\phi(H(i, gi)) + \phi(H(j, gj))\} \tag{28}$$

holds for every $i, j \in Y$ with $i \neq j$, then $g$ has a unique fixed point $i^*$ in $g$.

***Proof***

Let the sequence $\{i_n\}$ be defined as $i_{n+1} = gi_n$ for $n \in \mathbb{N}$. If $i_n = i_{n+1}$ for some $n$, then $g$ has a fixed point. So, let $i_n \neq i_{n+1}$ for every $n \in \mathbb{N}$.

Let $\alpha_n = \phi(H(i_n, i_{n+1}))$ for all $n \in \mathbb{N}$.

Therefore, it follows that

$$
\begin{aligned}
\alpha_{n+1} &= \phi(H(i_{n+1}, i_{n+2})) && (29)\\
&= \phi(H(gi_n, gi_{n+1})) && (30)\\
&\leq \frac{1}{2}\{\phi(H(i_n, gi_n)) + \phi(H(i_{n+1}, gi_{n+1}))\} && (31)\\
&= \frac{1}{2}\{\phi(H(i_n, i_{n+1})) + \phi(H(i_{n+1}, i_{n+2}))\} && (32)\\
&= \frac{1}{2}\{\alpha_n + \alpha_{n+1}\} && (33)\\
\alpha_{n+1} &< \frac{1}{2}\alpha_n + \frac{1}{2}\alpha_{n+1} && (34)\\
\alpha_{n+1} &< \alpha_n && (35)
\end{aligned}
$$

Hence,$\{\alpha_n\}$ is a strictly decreasing sequence of positive reals and hence converges to some nonnegative real number $a$.

Now, we claim that $a = 0$. If $a \neq 0$, then we have

$$o < a = \lim_{n\to\infty} \phi(H(i_n, i_{n+1})) = \phi(H(i^*, gi^*)) \tag{36}$$

which is a contradiction, hence $a = 0$.

Thus, the sequence$\{\alpha_n\}$ converges to zero.

Let there exists $(f, \alpha) \in F \times [0, \infty)$ satisfying the conditions $(H1 - H3)$ of definition 2.1. Then, by $(f2)$, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that.

$$0 < t < \delta \Rightarrow f(t) < f(\phi(\epsilon)) - \alpha \tag{37}$$

Now,

$$
\begin{aligned}
\phi(H(i_n, i_{n+1})) &< \frac{1}{2}\{\phi(H(i_{n-1}, gi_{n-1})) + \phi(H(i_n, gi_n))\} && (38)\\
&= \frac{1}{2}\{\phi(H(i_{n-1}, i_n)) + \phi(H(i_n, i_{n+1}))\} && (39)\\
&= \frac{1}{2}\{\phi(H(g^{n-1}i_0, g^n i_0)) + \phi(H(g^n i_0, g^{n+1} i_0))\} && (40)
\end{aligned}
$$

Similarly, we obtain,

$$\sum_{i=n}^{m-1} \phi(H(i_i, i_{i+1})) < \sum_{i=n}^{m-1} \frac{1}{2}\{\phi(H(g^{i-1}i_0, g^i i_0)) + \phi(H(g^i i_0, g^{i+1} i_0))\} \tag{41}$$

Since

$$\lim_{n\to\infty}\{\phi(H(g^{n-1}i_0, g^n i_0)) + \phi(H(g^n i_0, g^{n+1} i_0))\} = 0 \tag{42}$$

There exists some $N \in \mathbb{N}$ such that

$$0 < \sum_{i=n}^{m-1} \phi(H(i_i, i_{i+1})) < \delta$$

holds for all $n \geq N$. Hence by (37) and $(f1)$ we have

$$f\left(\sum_{i=n}^{m-1} \phi(H(i_i, i_{i+1}))\right) < f(\phi(\epsilon)) - \alpha \tag{43}$$

Now, we show that

$$H(i_n, i_m) < \epsilon \tag{44}$$

for all $m > n \geq \mathbb{N}$. Let $m, n \in \mathbb{N}$ be fixed but arbitrary such that $m > n \geq \mathbb{N}$. If$H(i_n, i_m) = 0$, then clearly $H(i_n, i_m) < \epsilon$ and if $H(i_n, i_m) > 0$, then using $(H3)$ and (43), we have

$$H(i_n, i_m) > 0 \tag{45}$$

$$\Rightarrow f\left(\phi(H(i_n, i_m))\right) \leq f\left(\sum_{i=n}^{m-1} \phi(H(i_i, i_{i+1}))\right) + \alpha < f(\phi(\epsilon)) \tag{46}$$

which gives by $(f1)$ that

$$(\phi(H(i_n, i_m))) < \phi(\epsilon) \tag{47}$$

$$\Rightarrow H(i_n, i_m) < \epsilon \tag{48}$$

This proves that $\{i_n\}$ is $F$-Cauchy. Since $\{i_n\}$ converges to $i_0$, then the limit of $\{i_n\}$ will be $i^*$. This implies.

$$\lim_{n\to\infty} H(i_n, i^*) = 0. \tag{49}$$

Since $gi_n = i_{n+1}$, we have, by the uniqueness of limit of sequence $i^* = gi^*$. Hence $i^*$ is a fixed point of $g$. For uniqueness, let $j^*$ be another fixed point of $g$. Then

$$\phi(H(j, j^*)) = \phi(H(gj, gj^*)) \tag{50}$$

$$\leq \frac{1}{2}\{\phi(H(j, gj)) + \phi(H(j^*, gj^*))\} \tag{51}$$

$$< 0 \tag{52}$$

a contradiction. Therefore $i^*$ is the unique fixed point of $g$.

**Theorem 3.3**

Let $(j, H)$ be an $F$-metric space and let $g : Y \to Y$ be a given mapping. Suppose that the following conditions are satisfied

i $(j, H)$ is $F$- complete,

ii there exists monotonically decreasing functions $a, b, c$ from $(0, \infty)$ into$[0, 1)$ satisfying $a(H(i, j))+b(H(i, j))+c(H(i, j)) < 1$ such that, for each $x, y \in X,\ x \neq y$

$$H(gi, gj) \leq a(H(i, j))H(i, gi) + b(H(i, j))H(j, gj) + c(H(i, j))H(i, j) \tag{53}$$

for all $(x, y) \in X \times Y$

then $g$ has a unique fixed point $i^* \in Y$. Moreover for any $i_0 \in Y$, the $\{i_n\} \in Y$ defined by $i_{n+1} = gi_n\ n \in \mathbb{N}$ is $F$-convergent to $i^*$.

***Proof***

By observation $g$ has at most one fixed point. Indeed, if $(u, v) \in X \times Y$ are two fixed point of $g$ with $u \neq v$ i.e $H(u, v) > 0\ \ gu = u\ \ and\ \ gv = v$

Then from (ii) we have

$$H(u, v) = H(gu, gv) \leq a(H(u, u))H(u, g) + b(H(u, v))H(v, gv) + c(H(u, v))H(u, v) \tag{54}$$
$$\leq c(H(u, v))H(u, v) \tag{55}$$

which is a contradiction.

Hence, $u = v$

Next, let $(f, \alpha) \in F \times [o, \infty)$ be such that (H3) is satisfied, let $\epsilon > 0$ be fixed.
$0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha$

Let $i_0 \in Y$ be an arbitrary element, let $\{i_n\} \in Y$ be sequence defined by $i_{n+1} = gi_n$ for $n \in \mathbb{N}$

$$H(gi_n, gi_{n+1}) \leq a(H(i, j))H(i_n, gi_n) + b(H(i, j))H(i_{n+1}, gi_{n+1}) \tag{56}$$
$$+ c(H(i, j))H(i_n, i_{n+1})H(i_{n+1}, i_{n+2}) \leq a(H(i, j))H(i_n, i_{n+1}) \tag{57}$$
$$+ b(H(i, j))H(i_{n+1}, i_{n+2}) + c(H(i, j))H(i_n, i_{n+1}) \tag{58}$$
$$H(i_{n+1}, i_{n+2}) - b(H(i, j))H(i_{n+1}, i_{n+2}) \leq [a(H(i, j)) + c(H(i, j))]H(i_n, i_{n+1}) \tag{59}$$
$$[1 - b(H(i, j))]H(i_{n+1}, i_{n+2}) \leq [a(H(i, j)) + c(H(i, j))]H(i_n, i_{n+1}) \tag{60}$$
$$H(i_{n+1}, i_{n+2}) \leq \frac{[a(H(i, j)) + c(H(i, j))]}{[1 - b(H(i, j))]}H(i_n, i_{n+1}) \tag{61}$$
$$let\ \ q = \frac{[a(H(i, j)) + c(H(i, j))]}{[1 - b(H(i, j))]} \tag{62}$$

then, $0 < q < 1$, since $a(H(i, j)) + b(H(i, j)) + c(H(i, j)) < 1$

Hence,

$$H(i_{n+1}, i_{n+2}) \leq qH(i_n, i_{n+1}) \tag{63}$$

Continue the above argument iteratively we will have

$$H(i_{n+1}, i_{n+2}) \leq \frac{q^n}{1 - q}H(i_0, i_1) \tag{64}$$

which yields

$$\sum_{i=n}^{m-1} H(i_i, i_{i+1}) \leq \frac{q^n}{1-q} H(i_0, i_1) \quad m > n \tag{65}$$

since $\lim\limits_{n\to\infty} \frac{q^n}{1-q} H(i_0, i_1) = 0$

there exists some $N \in \mathbb{N}$ such that

$$0 < \frac{q^n}{1-q} H(i_0, i_1) < \delta, \qquad n \geq \mathbb{N} \tag{66}$$

Hence, by $0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha$ and $(f1)$, we have

$$f\left(\sum_{i=n}^{m-1} H(i_i, i_{i+1})\right) \leq f\left(\frac{q^n}{1-q}(H(i_0, i_1))\right) < f(\epsilon) - \alpha \tag{67}$$

using (H3) and (67), we obtain

$$H(i_n, i_m) > 0, \quad m > n \geq N \Rightarrow f(H(i_n, i_m)) < f\left(\sum_{i=n}^{m-1} H(i_i, i_{i+1})\right) + \alpha < f(\epsilon) \tag{68}$$

which implies $H(i_n, i_m) < \epsilon, \quad m, n > N$

Therefore the $\{i_n\}$ if $F$-Cauchy, since $(j, H)$ is $F$-complete there exists $i^* \in Y$ such that $\{i_n\}$ is $F$-convergent to $i^*$ i.e

$$\lim_{n\to\infty} H(i_n, i^*) = 0 \tag{69}$$

we shall show that $i^*$ is a fixed point of $g$. We argue by contradiction by supposing that $H(gi^*, i^*) > 0$ . By (H3), we obtain

$$f(H(gi^*, i^*)) \leq f(H(gi^*, gi_n)) + f(H(gi_n, i^*)) + \alpha \tag{70}$$

using (ii) and $(f1)$, we obtain

$$f(H(gi^*, i^*)) \leq f(a(H(i,j))[H(i^*, i_n) + H(i_n, gi^*)] + b(H(i,j))[H(i^*, i_n) \tag{71}$$

$$+ H(i_n, gi^*)] + c(H(i,j))[H(i^*, i_n) + H(i_n, i^*)]) + \alpha \tag{72}$$

on the other hand, using $(f2)$ and $\lim\limits_{n\to\infty} H(i_n, i^*) = 0$ we have

$$\lim_{n\to\infty} f(a(H(i,j))[H(i^*, i_n) + H(i_n, gi^*)] + b(H(i,j))[H(i^*, i_n) + H(i_n, gi^*)] \tag{73}$$

$$+ c(H(i,j))[H(i^*, i_n) + H(i_n, i^*)]) + \alpha = -\infty \tag{74}$$

which is a contradiction, therefore, we have

$H(gi^*, i^*) = 0$ i.e $gi^* = i^*$. As a consequence, $i^* \in Y$ is the unique fixed point of $g$.

**Theorem 3.4**

Let $(j, H)$ be an $F$-metric space and let $g : Y \to Y$ be a given mapping. Suppose that the following conditions are satisfied.

i $(j, H)$ is $F$-complete,

ii for each $i, j \in Y \quad i \neq j$ such that

$$H(gi, gj) < h \max(H(i, gi), H(j, gj), H(i, j)) \tag{75}$$

Then $g$ has a unique fixed point $i^* \in Y$.

***Proof***

Let $i_0 \in Y$ be arbitrary but fixed and let $\{i_n\} \;\; n \geq 0$ be the Picard sequence of $g$ based on $i_0$, that is

$$i_{n+1} = gi_n \; for \;\; all \;\; n \geq 0 \tag{76}$$

If $i_n = i_{n+1}$ for some $n$, then it is easily noticeable that $i_n$ is a fixed point of $g$
Let $i_n \neq i_{n+1}$ for all $n \in \mathbb{N}$.

Putting $x = i_n$, $y = i_{n+1}$ in (75) and define sequence of a real number as $s_n = H(i_n, i_{n+1})$

$$H(gi_n, gi_{n+1}) < h \max(H(i_n, gi_n), H(i_{n+1}, gi_{n+1}) H(i_n, i_{n+1}), H(i_{n+1}, i_{n+2}),) \tag{77}$$
$$H(i_{n+1}, i_{n+2}) < h \max((i_n, i_{n+1}), H(i_{n+1}, i_{n+2}), H(i_n, i_{n+1})) \tag{78}$$
$$s_{n+1} < h \max(s_n, s_{n+1}, s_n) \tag{79}$$
$$s_{n+1} < h \max(s_n) \tag{80}$$
$$s_{n+1} < h(s_n) \tag{81}$$
$$\forall \;\; 0 < h < 1 \tag{82}$$

Therefore, $\{s_n\}$ is a monotone decreasing sequence of nonnegative real number.

Observe that $g$ has at most one fixed point if $(u, v) \in X \times Y$ are two fixed points of $g$ with $u \neq v$ i.e
$(u, v) > 0, \quad gu = u \;\; andgv = v$

then from equation(75) we have

$$H(u, v) < h \max(H(u, gu), H(v, gv), H(u, v)) \tag{83}$$
$$H(u, v) < h \max(H(u, v)) \tag{84}$$
$$H(u, v) < h(H(u, v)) \tag{85}$$

which is a contradiction hence, $u = v$.

Next, let $(f, \alpha) \in F \times [0, \infty)$ be such that (H3) is satisfied let $\epsilon > 0$ be fixed. By $(F2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha \tag{86}$$

let $i_0 \in Y$ be an arbitrary element, let $\{i_n\} \in Y$ be sequence defined by $i_{n+1} = gi_n \;\; n \in \mathbb{N}$

$$H(gi_n, gi_{n+1}) < h\max(H(i_n, gi_n), H(i_{n+1}, gi_{n+1})H(i_n, i_{n+1}), H(i_{n+1}, i_{n+2}),) \tag{87}$$
$$H(i_{n+1}, i_{n+2}) < h\max((i_n, i_{n+1}), H(i_{n+1}, i_{n+2}), H(i_n, i_{n+1})) \tag{88}$$

By (81) we have

$H(i_{n+1}, i_{n+2}) < H(i_n, i_{n+1})$

therefore

$$\begin{aligned} H(i_{n+1}, i_{n+2}) &< hH(i_n, i_{n+1}) && (89)\\ &= hH(gi_{n-1}, gi_n) && (90)\\ &\le h^2 H(i_{n-1}, i_n) && (91)\\ &= h^2 H(gi_{n-2}, gi_{n-}) && (92)\\ &\le h^3 H(i_{n-2}, i_{n-1}) && (93)\\ & && (94)\end{aligned}$$

consequently, by induction for all $n \in \mathbb{N}$, we have

$H(i_{n+1}, i_{n+2}) < h^n H(i_0, i_1)$ which yield

$$\sum_{i=n}^{m-1} H(i_i, i_{i+1}) \le \frac{h^n}{1-h} H(i_0, i_1) \quad m > n \tag{95}$$

since $\lim\limits_{n\to\infty} \frac{h^n}{1-h} H(i_0, i_1) = 0$

there exists some $N \in \mathbb{N}$ such that

$$0 < \frac{h^n}{1-h} H(i_0, i_1) < \delta, \qquad n \ge \mathbb{N} \tag{96}$$

Hence, by $0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha$ and $(f1)$, we have

$$f\left(\sum_{i=n}^{m-1} H(i_i, i_{i+1})\right) \le f\left(\frac{q^n}{1-q}(H(i_0, i_1))\right) < f(\epsilon) - \alpha \tag{97}$$

using (H3) and (97), we obtain

$$H(i_n, i_m) > 0, \quad m > n \ge N \Rightarrow f(H(i_n, i_m)) < f\left(\sum_{i=n}^{m-1} H(i_i, i_{i+1})\right) + \alpha < f(\epsilon) \tag{98}$$

which implies $H(i_n, i_m) < \epsilon, \quad m, n > N$

Therefore the $\{i_n\}$ if $F$-Cauchy, since $(j, H)$ is $F$-complete there exists $i^* \in Y$ such that $\{i_n\}$ is $F$-convergent to $i^*$ i.e

$$\lim_{n\to\infty} H(i_n, i^*) = 0 \tag{99}$$

which shows that $\{i_n\}$ is $F$-Cauchy.

Hence, $F$-completeness of $(j, H)$ implies that there exists $i^* \in Y$ such that $i_n \to i^*$ as we shall proof that $i^*$ is a fixed point of $g$. We argue by contradiction by supposing that $H(gi^*, i^*) > 0$. By (H3) we have

$$f(H(gi^*, i^*)) \leq f(H(gi^*, gi_n)) + f(H(gi_n, i^*)) + \alpha \tag{100}$$

using(75) and $(f1)$, we have.

$$f(H(gi^*, i^*)) \leq f(h \max[H(i^*, i_n) + H(i_n, gi^*)], [H(i^*, i_{n+1}) + H(i^*, i_{n+1})], \tag{101}$$

$$[H(i^*, i_{n+1}) + H(i_n, i^*)]) + \alpha \tag{102}$$

on the other hand using $(f2)$ and (99), we have

$$\lim_{n\to\infty} f(H(gi^*, i^*)) \leq f(h \max[H(i^*, i_n) + H(i_n, gi^*)], [H(i^*, i_{n+1}) + H(i^*, i_{n+1})], \tag{103}$$

$$[H(i^*, i_{n+1}) + H(i_n, i^*)]) + \alpha = -\infty \tag{104}$$

which is a contradiction, therefore, we have

$H(gi^*, i^*) = 0$ i.e $gi^* = i^*$. As a consequence, $i^* \in Y$ is the unique fixed point of $g$.

# 4 Conclusion

The article established the existence and uniqueness of fixed point in the setting of F-metric spaces. We considered p-contraction, monotonically decreasing and some other contractions to established our facts. The work extended and generalized several results in the literature.

## Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

# Competing Interests

Authors have declared that no competing interests exist.

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