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Generalized Stability of a General Quintic Functional Equation

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The general quintic functional equation extends the framework of numerous classical functional equations, including Jensen, quadratic, cubic, and quartic equations, offering a unified perspective on their stability. This paper investigates the generalized stability of the quintic functional equation using advanced mathematical techniques, including the direct method and rigorous computational analysis. By providing improved and concise proofs, this study enhances existing stability results and extends their applicability under broader conditions. These findings contribute to the theoretical foundations of functional equations, with potential implications for diverse areas in mathematics and its applications.

Keywords: Stability of a functional equation; general quintic functional equation; general quintic mapping.

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1 Introduction

In this paper, let V, X, and Y be a real vector space, a real normed space, and a real Banach space, respectively. The result of the stability of additive functional equation obtained by Hyers (1941) as an answer to an question of group isomorphism raised by Ulam (1960) in 1940 became the starting point for stability of functional equations, and many mathematicians followed him to study the stability of various types of functional equations (see Găvruta (1994); Rassias (1978) for more generalized results).

Consider the general quintic functional equation

$$\sum_{i=0}^{6}\binom{6}{i}(-1)^{6-i}f(x+iy)=0 \tag{1.1}$$

for all $x, y \in V$. If $f : V \to Y$ is a solution mapping of the functional equation (1.1), then we call the mapping f a general quintic mapping. The result obtained by Y. H. Lee for the Hyers-Ulam-Rassas stability of the function equation (1.1) is shown in the following theorem.

Theorem 1.1. (Theorem 2 in Lee (2019b)) *Let $p \neq 1,2,3,4,5$ be a fixed nonnegative real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\left\|\sum_{i=0}^{6}{}_6C_i(-1)^{6-i}f(x+iy)\right\| \le \theta(\|x\|^p+\|y\|^p) \tag{1.2}$$

for all $x, y \in X$. Then there exists a general quintic mapping F with $F(0) = 0$ and a constant $K(p)$ such that

$$\|f(x)-f(0)-F(x)\| \le K(p)\theta\|x\|^p$$

for all $x \in X$.

The hyperstability of the functional equation (1.1) obtained by S. S. Jin and Y. H. Lee is as follows.

Theorem 1.2. (Theorem 2.4 in Jin and Lee (2023)) *Let $p < 0$ be a real number. Suppose that $f : X \to Y$ is a mapping satisfying the inequality* (1.2) *for all $x, y \in X \setminus \{0\}$. Then f satisfies the functional* (1.1).

Lee and Jung (2023) obtained partial results of the generalized stability of the functional equation (1.1) using the fixed point method. On the other hand Jin and Lee (2021) used the method of P. Găvruta in (Găvruta, 1994) to obtain partial stability results of (1.1), too.

In this paper, we will show concise results that have improved the existing results on the stability of the general quintic functional equation in the spirit of P. Găvruta through a clearer proof. In particular, we will extend the range of partial results of the generalized stability of the functional equation (1.1) obtained by Lee and Jung (2023) and Jin and Lee (2021) to general results. For research on the stability of the general cubic functional equation and the stability of general quartic functional equation, which correspond to prior research on the stability of the general quintic functional equation, see the studies of Lee and Jung (2020), Jin and Lee (2024), Jun and Kim (2003), and Lee (2019a).

2 Stability of a General Quintic Functional Equation

Throughout this paper, for a given mapping $f : V \to Y$, we use the following abbreviations:

$$\begin{aligned}
&\tilde{f}(x) := f(x) - f(0),\\
&f_1(x) := \frac{1}{5040}\big(\tilde{f}(16x) - 60\tilde{f}(8x) + 1120\tilde{f}(4x) - 7680\tilde{f}(2x) + 16384\tilde{f}(x)\big),\\
&f_2(x) := -\frac{1}{2688}\big(\tilde{f}(16x) - 58\tilde{f}(8x) + 1008\tilde{f}(4x) - 5888\tilde{f}(2x) + 8192\tilde{f}(x)\big),\\
&f_3(x) := \frac{1}{4608}\big(\tilde{f}(16x) - 54\tilde{f}(8x) + 808\tilde{f}(4x) - 3456\tilde{f}(2x) + 4096\tilde{f}(x)\big),\\
&f_4(x) := -\frac{1}{21504}\big(\tilde{f}(16x) - 46\tilde{f}(8x) + 504\tilde{f}(4x) - 1856\tilde{f}(2x) + 2048\tilde{f}(x)\big),\\
&f_5(x) := \frac{1}{322560}\big(\tilde{f}(16x) - 30\tilde{f}(8x) + 280\tilde{f}(4x) - 960\tilde{f}(2x) + 1024\tilde{f}(x)\big),\\
&\mathop{\Delta}^{6}_{y} f(x) := \sum_{i=0}^{6}\binom{6}{i}(-1)^{6-i} f(x+iy),\\
&\Gamma f(x) := f(32x) - 62f(16x) + 1240f(8x) - 9920f(4x) + 31744f(2x) - 32768f(x)
\end{aligned}$$

for all $x, y \in V$. By laborious computation we can get some useful equalities in the following lemma.

Lemma 2.1. *For a given mapping $f : V \to Y$, the equalities*

$$\mathop{\Delta}^{6}_{y} \tilde{f}(x) = \mathop{\Delta}^{6}_{y} f(x),$$

$$\begin{aligned}
\Gamma\tilde{f}(x) = {} & \mathop{\Delta}^{6}_{4x} f(8x) + 6\mathop{\Delta}^{6}_{4x} f(4x) + 21\mathop{\Delta}^{6}_{-4x} f(24x) + 56\mathop{\Delta}^{6}_{2x} f(8x) + 336\mathop{\Delta}^{6}_{2x} f(6x)\\
& + 904\mathop{\Delta}^{6}_{2x} f(4x) + 1504\mathop{\Delta}^{6}_{2x} f(2x) + 1680\mathop{\Delta}^{6}_{-2x} f(12x) + 896\mathop{\Delta}^{6}_{x} f(4x)\\
& + 5376\mathop{\Delta}^{6}_{x} f(3x) + 13056\mathop{\Delta}^{6}_{x} f(2x) + 15616\mathop{\Delta}^{6}_{x} f(x) + 8064\mathop{\Delta}^{6}_{-x} f(6x),
\end{aligned} \tag{2.1}$$

$$\tilde{f}_1(x) - \frac{\tilde{f}_1(2x)}{2} = -\frac{\Gamma\tilde{f}(x)}{10080}, \qquad \tilde{f}_1(x) - 2\tilde{f}_1\Big(\frac{x}{2}\Big) = \frac{1}{5040}\Gamma\tilde{f}\Big(\frac{x}{2}\Big), \tag{2.2}$$

$$\tilde{f}_2(x) - \frac{\tilde{f}_2(2x)}{4} = \frac{\Gamma\tilde{f}(x)}{10752}, \qquad \tilde{f}_2(x) - 4\tilde{f}\Big(\frac{x}{2}\Big) = -\frac{1}{2688}\Gamma\tilde{f}\Big(\frac{x}{2}\Big), \tag{2.3}$$

$$\tilde{f}_3(x) - \frac{\tilde{f}_3(2x)}{8} = -\frac{\Gamma\tilde{f}(x)}{36864}, \qquad \tilde{f}_3(x) - 8\tilde{f}_3\Big(\frac{x}{2}\Big) = \frac{1}{4608}\Gamma\tilde{f}\Big(\frac{x}{2}\Big), \tag{2.4}$$

$$\tilde{f}_4(x) - \frac{\tilde{f}_4(2x)}{16} = \frac{\Gamma\tilde{f}(x)}{344064}, \qquad \tilde{f}_4(x) - 16\tilde{f}_4\Big(\frac{x}{2}\Big) = -\frac{1}{21504}\Gamma\tilde{f}\Big(\frac{x}{2}\Big), \tag{2.5}$$

$$\tilde{f}_5(x) - \frac{\tilde{f}_5(2x)}{32} = -\frac{\Gamma\tilde{f}(x)}{10321920}, \qquad \tilde{f}_5(x) - 32\tilde{f}_5\Big(\frac{x}{2}\Big) = \frac{1}{322560}\Gamma\tilde{f}\Big(\frac{x}{2}\Big), \tag{2.6}$$

$$f(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x) \tag{2.7}$$

hold for all $x, y \in V$.

Lemma 2.2. *If $f : V \to Y$ satisfies the functional equation $\mathop{\Delta}^{6}_{y} f(x) = 0$ for all $x, y \in V$, then the mappings $\tilde{f}_1, \ldots, \tilde{f}_5 : V \to Y$ satisfies*

$$\tilde{f}_k(2x) = 2^k \tilde{f}_k(x) \tag{2.8}$$

for all $x \in V$ and each $k \in \{1,2,3,4,5\}$.

Proof. If $f: V \to Y$ satisfies the functional equation $\overset{6}{\underset{y}{\Delta}} f(x) = 0$ for all $x, y \in V$, then $f: V \to Y$ satisfies the functional equation $\Gamma\tilde{f}(x) = 0$ from (2.1). Therefore, the equality (2.8) follows from the equalities (2.2), (2.3), (2.4), (2.5), and (2.6). □

According to Corollary 6 in Jung et al., we obtain following Lemma.

Lemma 2.3. *For a given mapping $f: V \to Y$, if there exist a mapping $F: V \to Y$ and a function $\phi: V \setminus \{0\} \to [0,\infty)$ that satisfy*

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{1}{2^i}\phi\left(2^i x\right) < \infty \ \ or \tag{2.9}$$

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{1}{2^{(\ell+1)i}}\phi\left(2^i x\right) + \sum_{i=0}^{\infty} 2^{\ell i}\phi\left(\frac{1}{2^i}x\right) < \infty \ \ or \tag{2.10}$$

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} 2^{5i}\phi\left(\frac{1}{2^i}x\right) < \infty \tag{2.11}$$

for all $x \in V \setminus \{0\}$ and for some $\ell \in \{1,2,3,4\}$, where $F(x) = \sum_{k=1}^{5} F_k(x)$ and every F_k has the property (2.8)*, then the mapping F is uniquely determined.*

Lemma 2.4. *If a mapping $f: V \to Y$ satisfies the functional equation $\overset{6}{\underset{y}{\Delta}} f(x) = 0$ for all $x, y \in V \setminus \{0\}$, then it is a general quintic mapping.*

Proof. It is clear that $\overset{6}{\underset{0}{\Delta}} f(x) = 0$ for all $x \in V$ and

$$\overset{6}{\underset{y}{\Delta}} f(0) = \overset{6}{\underset{-y}{\Delta}} f(6y) = 0$$

for all $y \in V \setminus \{0\}$. So $\overset{6}{\underset{y}{\Delta}} f(x) = 0$ for all $x, y \in V$ as desired. □

Now we show the generalized stability theorem of (1.1).

Theorem 2.5. *Let $\varphi: (V \setminus \{0\})^2 \to [0,\infty)$ be a function satisfying one of the following conditions*

$$\sum_{i=0}^{\infty} 2^{-i}\varphi(2^i x, 2^i y) < \infty, \tag{2.12}$$

$$\sum_{i=0}^{\infty} 4^{-i}\varphi(2^i x, 2^i y) < \infty \ \ and \ \ \sum_{i=0}^{\infty} 2^i\varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.13}$$

$$\sum_{i=0}^{\infty} 8^{-i}\varphi(2^i x, 2^i y) < \infty \ \ and \ \ \sum_{i=0}^{\infty} 4^i\varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.14}$$

$$\sum_{i=0}^{\infty} 16^{-i}\varphi(2^i x, 2^i y) < \infty \ \ and \ \ \sum_{i=0}^{\infty} 8^i\varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.15}$$

$$\sum_{i=0}^{\infty} 32^{-i}\varphi(2^i x, 2^i y) < \infty \ \ and \ \ \sum_{i=0}^{\infty} 16^i\varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.16}$$

$$\sum_{i=0}^{\infty} 32^i\varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{2.17}$$

for all $x, y \in V\setminus\{0\}$. Suppose that $f : V \to Y$ is a mapping such that

$$\left\| \overset{6}{\underset{y}{\Delta}} f(x) \right\| \le \varphi(x,y) \tag{2.18}$$

for all $x, y \in V\setminus\{0\}$. Then there exists a unique general quintic mapping F such that

$$\|\tilde{f}(x) - F(x)\| \le \frac{1}{10080}\sum_{i=0}^{\infty}\frac{\Phi(2^i x)}{2^i}, \tag{2.19}$$

$$\|\tilde{f}(x) - F(x)\| \le \sum_{i=0}^{\infty}\frac{2^i}{5040}\Phi\Big(\frac{x}{2^{i+1}}\Big) + \sum_{i=0}^{\infty}\frac{\Phi(2^i x)}{10752\cdot 4^i}, \tag{2.20}$$

$$\|\tilde{f}(x) - F(x)\| \le \sum_{i=0}^{\infty}\frac{4^i}{2688}\Phi\Big(\frac{x}{2^{i+1}}\Big) + \sum_{i=0}^{\infty}\frac{\Phi(2^i x)}{36864\cdot 8^i}, \tag{2.21}$$

$$\|\tilde{f}(x) - F(x)\| \le \sum_{i=0}^{\infty}\frac{8^i}{4608}\Phi\Big(\frac{x}{2^{i+1}}\Big) + \sum_{i=0}^{\infty}\frac{\Phi(2^i x)}{344064\cdot 16^i}, \tag{2.22}$$

$$\|\tilde{f}(x) - F(x)\| \le \sum_{i=0}^{\infty}\frac{16^i}{21504}\Phi\Big(\frac{x}{2^{i+1}}\Big) + \sum_{i=0}^{\infty}\frac{\Phi(2^i x)}{10321920\cdot 32^i}, \tag{2.23}$$

$$\|\tilde{f}(x) - F(x)\| \le \sum_{i=4}^{\infty}\frac{32^i}{322560}\Phi\Big(\frac{x}{2^{i+1}}\Big) \tag{2.24}$$

for all $x \in V\setminus\{0\}$ if φ satisfies (2.12), (2.13), (2.14), (2.15), (2.16), or (2.17), respectively, where $\Phi : V\setminus\{0\} \to [0,\infty)$ is the function defined by

$$\begin{aligned}\Phi(x) :=& \varphi(8x,4x) + 6\varphi(4x,4x) + 21\varphi(24x,-4x) + 56\varphi(8x,2x) + 336\varphi(6x,2x)\\ &+ 904\varphi(4x,2x) + 1504\varphi(2x,2x) + 1680\varphi(12x,-2x) + 896\varphi(4x,x)\\ &+ 5376\varphi(3x,x) + 13056\varphi(2x,x) + 15616\varphi(x,x) + 8064\varphi(6x,-x).\end{aligned}$$

Proof. Notice that, from(2.1) and (2.18), we have

$$\begin{aligned}\|\Gamma\tilde{f}(x)\| =& \Big\| \overset{6}{\underset{4x}{\Delta}} f(8x) + 6\overset{6}{\underset{4x}{\Delta}} f(4x) + 21\overset{6}{\underset{-4x}{\Delta}} f(24x) + 56\overset{6}{\underset{2x}{\Delta}} f(8x) + 336\overset{6}{\underset{2x}{\Delta}} f(6x)\\ &+ 904\overset{6}{\underset{2x}{\Delta}} f(4x) + 1504\overset{6}{\underset{2x}{\Delta}} f(2x) + 1680\overset{6}{\underset{-2x}{\Delta}} f(12x) + 896\overset{6}{\underset{x}{\Delta}} f(4x)\\ &+ 5376\overset{6}{\underset{x}{\Delta}} f(3x) + 13056\overset{6}{\underset{x}{\Delta}} f(2x) + 15616\overset{6}{\underset{x}{\Delta}} f(x) + 8064\overset{6}{\underset{-x}{\Delta}} f(6x)\Big\|\\ \le& \Phi(x)\end{aligned} \tag{2.25}$$

for all $x \in V$. We prove the theorem in two steps.

Step 1. Let $k \in \{1,2,3,4,5\}$ and $\delta \in \{-1,1\}$, and let φ satisfy

$$\sum_{n=0}^{\infty}\frac{\varphi(2^{\delta n}x, 2^{\delta n}y)}{2^{\delta kn}} < \infty \tag{2.26}$$

for all $x, y \in V\setminus\{0\}$. Together with

$$\frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta kn}} - \frac{\tilde{f}_k(2^{\delta(n+m)}x)}{2^{\delta k(n+m)}} = \sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_k(2^{\delta i}x)}{2^{\delta ki}} - \frac{\tilde{f}_k(2^{\delta(i+1)}x)}{2^{\delta k(i+1)}}\right)$$

and (2.2), (2.3), (2.4), (2.5), (2.6), (2.25), we have the inequalities

$$\left\|\frac{\tilde{f}_1(2^n x)}{2^n} - \frac{\tilde{f}_1(2^{n+m}x)}{2^{n+m}}\right\| \le \frac{1}{10080}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{2^i}\right\| \le \frac{1}{10080}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{2^i},$$

$$\left\|2^n\tilde{f}_1\left(\frac{x}{2^n}\right) - 2^{n+m}\tilde{f}_1\left(\frac{x}{2^{n+m}}\right)\right\| \le \frac{1}{5040}\sum_{i=n}^{n+m-1}\left\|2^i\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \le \frac{1}{5040}\sum_{i=n}^{n+m-1}2^i\Phi\left(\frac{x}{2^{i+1}}\right),$$

$$\left\|\frac{\tilde{f}_2(2^n x)}{4^n} - \frac{\tilde{f}_2(2^{n+m}x)}{4^{n+m}}\right\| \le \frac{1}{10752}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{4^i}\right\| \le \frac{1}{10752}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{4^i},$$

$$\left\|4^n\tilde{f}_2\left(\frac{x}{2^n}\right) - 4^{n+m}\tilde{f}_2\left(\frac{x}{2^{n+m}}\right)\right\| \le \frac{1}{2688}\sum_{i=n}^{n+m-1}\left\|4^i\,\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \le \frac{1}{2688}\sum_{i=n}^{n+m-1}4^i\Phi\left(\frac{x}{2^{i+1}}\right),$$

$$\left\|\frac{\tilde{f}_3(2^n x)}{8^n} - \frac{\tilde{f}_3(2^{n+m}x)}{8^{n+m}}\right\| \le \frac{1}{36864}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{8^i}\right\| \le \frac{1}{36864}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{8^i},$$

$$\left\|8^n\tilde{f}_3\left(\frac{x}{2^n}\right) - 8^{n+m}\tilde{f}_3\left(\frac{x}{2^{n+m}}\right)\right\| \le \frac{1}{4608}\sum_{i=n}^{n+m-1}\left\|8^i\,\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \le \frac{1}{4608}\sum_{i=n}^{n+m-1}8^i\Phi\left(\frac{x}{2^{i+1}}\right),$$

$$\left\|\frac{\tilde{f}_4(2^n x)}{16^n} - \frac{\tilde{f}_4(2^{n+m}x)}{16^{n+m}}\right\| \le \frac{1}{344064}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{16^i}\right\| \le \frac{1}{344064}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{16^i},$$

$$\begin{aligned}\left\|16^n\tilde{f}_3\left(\frac{x}{2^n}\right) - 16^{n+m}\tilde{f}_3\left(\frac{x}{2^{n+m}}\right)\right\| &\le \frac{1}{21504}\sum_{i=n}^{n+m-1}\left\|16^i\,\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \\ &\le \frac{1}{21504}\sum_{i=n}^{n+m-1}16^i\Phi\left(\frac{x}{2^{i+1}}\right),\end{aligned}$$

$$\left\|\frac{\tilde{f}_5(2^n x)}{32^n} - \frac{\tilde{f}_5(2^{n+m}x)}{32^{n+m}}\right\| \le \frac{1}{10321920}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{32^i}\right\| \le \frac{1}{10321920}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{32^i},$$

and

$$\begin{aligned}\left\|32^n\tilde{f}_5\left(\frac{x}{2^n}\right) - 32^{n+m}\tilde{f}_5\left(\frac{x}{2^{n+m}}\right)\right\| &\le \frac{1}{322560}\sum_{i=n}^{n+m-1}\left\|32^i\,\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \\ &\le \frac{1}{322560}\sum_{i=n}^{n+m-1}32^i\Phi\left(\frac{x}{2^{i+1}}\right)\end{aligned}$$

for all $x \in V\backslash\{0\}$ and $n, m \in \mathbb{N} \cup \{0\}$. It leads us to prove that $\left\{\frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta kn}}\right\}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$ if φ satisfies (2.26). Moreover, since Y is complete and $\tilde{f}_k(0) = 0$, the sequence converges for all $x \in V$. It follows that we can define a mapping $F_{\delta k} : V \to Y$ by

$$F_{\delta k}(x) := \lim_{n\to\infty} \frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta kn}} \tag{2.27}$$

for all $x \in V$ if φ satisfies (2.26). Now we observe that the equality

$$\begin{aligned}
\overset{6}{\underset{y}{\Delta}} F_{\delta k}(x) =& F_{\delta k}(x+6y) - 6F_{\delta k}(x+5y) + 15F_{\delta k}(x+4y) - 20F_{\delta k}(x+3y) \\
&+ 15F_{\delta k}(x+2y) - 6F_{\delta k}(x+y) + F_{\delta k}(x) \\
=& \lim_{n\to\infty}\Big(\frac{\tilde{f}_k(2^{\delta n}(x+6y))}{2^{\delta kn}} - 6\frac{\tilde{f}_k(2^{\delta n}(x+5y))}{2^{\delta kn}} + 15\frac{\tilde{f}_k(2^{\delta n}(x+4y))}{2^{\delta kn}} \\
&- 20\frac{\tilde{f}_k(2^{\delta n}(x+3y))}{2^{\delta kn}} + 15\frac{\tilde{f}_k(2^{\delta n}(x+2y))}{2^{\delta kn}} - 6\frac{\tilde{f}_k(2^{\delta n}(x+y))}{2^{\delta kn}} + \frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta kn}}\Big)
\end{aligned}$$

holds for all $x, y \in V\backslash\{0\}$. Together with the definition of $\tilde{f}_1$, if φ satisfies (2.26) for $k = 1$, then we have

$$\begin{aligned}
\left\|\overset{6}{\underset{y}{\Delta}} F_{\delta 2}(x)\right\| =& \lim_{n\to\infty}\Big\|\frac{\Delta^6_{2^{\delta n+4}y} f(2^{\delta n+4}x)}{-2688\cdot 4^{\delta n}} + \frac{58\Delta^6_{2^{\delta n+3}y} f(2^{\delta n+3}x)}{2688\cdot 4^{\delta n}} - \frac{1088\Delta^6_{2^{\delta n+2}y} f(2^{\delta n+2}x)}{2688\cdot 4^{\delta n}} \\
&+ \frac{5888\Delta^6_{2^{\delta n+1}y} f(2^{\delta n+1}x)}{2688\cdot 4^{\delta n}} - \frac{8192\,\Delta^6_{2^{\delta n}y} f(2^{\delta n}x)}{2688\cdot 4^{\delta n}}\Big\| \\
\le& \lim_{n\to\infty}\Big(\frac{\varphi(2^{\delta n+4}x, 2^{\delta n+4}y)}{2688\cdot 4^{\delta n}} + \frac{58\varphi(2^{\delta n+3}x, 2^{\delta n+3}y)}{2688\cdot 4^{\delta n}} + \frac{1088\varphi(2^{\delta n+2}x, 2^{\delta n+2}y)}{2688\cdot 4^{\delta n}} \\
&+ \frac{5888\varphi(2^{\delta n+1}x, 2^{\delta n+1}y)}{2688\cdot 4^{\delta n}} + \frac{8192\varphi(2^{\delta n}x, 2^{\delta n}y)}{2688\cdot 4^{\delta n}}\Big) \\
=& 0,
\end{aligned}$$

$$\begin{aligned}
\left\|\overset{6}{\underset{y}{\Delta}} F_{\delta 3}(x)\right\| =& \lim_{n\to\infty}\Big\|\frac{\Delta^6_{2^{\delta n+4}y} f(2^{\delta n+4}x)}{4608\cdot 8^{\delta n}} - \frac{54\Delta^6_{2^{\delta n+3}y} f(2^{\delta n+3}x)}{4608\cdot 8^{\delta n}} + \frac{808\Delta^6_{2^{\delta n+2}y} f(2^{\delta n+2}x)}{4608\cdot 8^{\delta n}} \\
&- \frac{3456\Delta^6_{2^{\delta n+1}y} f(2^{\delta n+1}x)}{4608\cdot 8^{\delta n}} + \frac{4096\,\Delta^6_{2^{\delta n}y} f(2^{\delta n}x)}{4608\cdot 8^{\delta n}}\Big\| \\
\le& \lim_{n\to\infty}\Big(\frac{\varphi(2^{\delta n+4}x, 2^{\delta n+4}y)}{4608\cdot 8^{\delta n}} + \frac{54\varphi(2^{\delta n+3}x, 2^{\delta n+3}y)}{4608\cdot 8^{\delta n}} + \frac{808\varphi(2^{\delta n+2}x, 2^{\delta n+2}y)}{4608\cdot 8^{\delta n}} \\
&+ \frac{3456\varphi(2^{\delta n+1}x, 2^{\delta n+1}y)}{4608\cdot 8^{\delta n}} + \frac{4096\varphi(2^{\delta n}x, 2^{\delta n}y)}{4608\cdot 8^{\delta n}}\Big) \\
=& 0,
\end{aligned}$$

$$\begin{aligned}\left\|\overset{6}{\underset{y}{\Delta}}F_{\delta 4}(x)\right\| &= \lim_{n\to\infty}\Bigg\|\frac{\Delta^6_{2^{\delta n+4}y}f(2^{\delta n+4}x)}{-21504\cdot 16^{\delta n}}+\frac{46\Delta^6_{2^{\delta n+3}y}f(2^{\delta n+3}x)}{21504\cdot 16^{\delta n}}-\frac{504\Delta^6_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{21504\cdot 16^{\delta n}}\\ &\quad+\frac{1856\Delta^6_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{21504\cdot 16^{\delta n}}-\frac{2048\Delta^6_{2^{\delta n}y}f(2^{\delta n}x)}{21504\cdot 16^{\delta n}}\Bigg\|\\ &\le \lim_{n\to\infty}\Bigg(\frac{\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{21504\cdot 16^{\delta n}}+\frac{46\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{21504\cdot 16^{\delta n}}+\frac{504\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{21504\cdot 16^{\delta n}}\\ &\quad+\frac{1856\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{21504\cdot 16^{\delta n}}+\frac{2048\varphi(2^{\delta n}x,2^{\delta n}y)}{21504\cdot 16^{\delta n}}\Bigg)\\ &= 0,\end{aligned}$$

$$\begin{aligned}\left\|\overset{6}{\underset{y}{\Delta}}F_{\delta 5}(x)\right\| &= \lim_{n\to\infty}\Bigg\|\frac{\Delta^6_{2^{\delta n+4}y}f(2^{\delta n+4}x)}{322560\cdot 32^{\delta n}}-\frac{30\Delta^6_{2^{\delta n+3}y}f(2^{\delta n+3}x)}{322560\cdot 32^{\delta n}}+\frac{280\Delta^6_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{322560\cdot 32^{\delta n}}\\ &\quad-\frac{960\Delta^6_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{322560\cdot 32^{\delta n}}+\frac{1024\Delta^6_{2^{\delta n}y}f(2^{\delta n}x)}{322560\cdot 32^{\delta n}}\Bigg\|\\ &\le \lim_{n\to\infty}\Bigg(\frac{\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{322560\cdot 32^{\delta n}}+\frac{30\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{322560\cdot 32^{\delta n}}+\frac{280\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{322560\cdot 8^{\delta n}}\\ &\quad+\frac{960\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{322560\cdot 32^{\delta n}}+\frac{1024\varphi(2^{\delta n}x,2^{\delta n}y)}{322560\cdot 32^{\delta n}}\Bigg)\\ &= 0\end{aligned}$$

for all $x,y\in V\setminus\{0\}$. And then, since $\Delta^6_y F_{\delta k}(x)=0$ for all $x,y\in V\setminus\{0\}$, the mapping $F_{\delta k}$ is a general quintic mapping for all $k\in\{1,2,3,4,5\}$ and $\delta\in\{+1,-1\}$ by Lemma 2.4.

Step 2. Now we define the desired general quintic mapping F for all cases.

(1) Let φ satisfy the condition (2.12), then F_1, F_2, F_3, F_4, and F_5 are defined by (2.27). We put a general quintic mapping $F: V\to Y$ by

$$F(x) := F_1(x)+F_2(x)+F_3(x)+F_4(x)+F_5(x)$$

for all $x\in V$. Observe that. by (2.2), (2.3), (2.4), (2.5), and (2.6), we have

$$\begin{aligned}\left\|\tilde f(x)-\sum_{k=1}^{5}\frac{\tilde f_k(2^n x)}{2^{kn}}\right\| &\le \sum_{i=0}^{n-1}\left\|\sum_{k=1}^{5}\left(\frac{\tilde f_k(2^i x)}{2^{ki}}-\frac{\tilde f_k(2^{i+1}x)}{2^{k(i+1)}}\right)\right\|\\ &= \sum_{i=0}^{n-1}\left(\frac{1}{10080\cdot 2^i}-\frac{1}{10752\cdot 4^i}+\frac{1}{36864\cdot 8^i}-\frac{1}{344064\cdot 16^i}+\frac{1}{10321920\cdot 32^i}\right)\|\Gamma\tilde f(2^i x)\|\\ &\le \sum_{i=0}^{n-1}\left\|\frac{\Gamma\tilde f(2^i x)}{10080\cdot 2^i}\right\| \le \frac{1}{10080}\sum_{i=0}^{n-1}\frac{\Phi(2^i x)}{2^i}\end{aligned}$$

for all $x\in V\setminus\{0\}$, which follows (2.19) as $n\to\infty$.

(2) Let φ satisfy the condition (2.13), then F_{-1}, F_2, F_3, F_4, and F_5 are defined by (2.27). Putting a general quintic mapping $F: V\to Y$ by

$$F(x) := F_{-1}(x)+F_2(x)+F_3(x)+F_4(x)+F_5(x)$$

for all $x \in V$. Then we have

$$\begin{aligned}
&\left\| \tilde{f}(x) - 2^n \tilde{f}_1\left(\frac{x}{2^n}\right) - \sum_{k=2}^{5} \frac{\tilde{f}_k(2^n x)}{2^{kn}} \right\| \\
&\le \sum_{i=0}^{n-1} \left\| 2^i \tilde{f}_1\left(\frac{x}{2^i}\right) - 2^{i+1} \tilde{f}_1\left(\frac{x}{2^{i+1}}\right) \right\| + \sum_{i=0}^{n-1} \left\| \sum_{k=2}^{5} \left(\frac{\tilde{f}_k(2^i x)}{2^{ki}} - \frac{\tilde{f}_k(2^{i+1}x)}{2^{k(i+1)}} \right) \right\| \\
&\le \sum_{i=0}^{n-1} \frac{2^i}{5040} \left\| \Gamma \tilde{f}\left(\frac{x}{2^{i+1}}\right) \right\| \\
&\quad + \sum_{i=0}^{n-1} \left(\frac{1}{10752 \cdot 4^i} - \frac{1}{36864 \cdot 8^i} + \frac{1}{344064 \cdot 16^i} - \frac{1}{10321920 \cdot 32^i} \right) \|\Gamma \tilde{f}(2^i x)\| \\
&\le \frac{1}{5040} \sum_{i=0}^{n-1} 2^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{10752} \sum_{i=0}^{n-1} \frac{\Phi(2^i x)}{4^i}
\end{aligned}$$

for all $x \in V \setminus \{0\}$ by (2.2), (2.3), (2.4), (2.5), and (2.6), which follows (2.20) as $n \to \infty$.

(3) Let φ satisfy the condition (2.14), then F_{-1}, F_{-2}, F_3, F_4, and F_5 are defined by (2.27). Putting a general quintic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_3(x) + F_4(x) + F_5(x)$$

for all $x \in V$. We have the inequality

$$\begin{aligned}
&\left\| \tilde{f}(x) - \sum_{k=1}^{2} 2^{kn} \tilde{f}_k\left(\frac{x}{2^n}\right) - \sum_{k=3}^{5} \frac{\tilde{f}_k(2^n x)}{2^{kn}} \right\| \\
&\le \sum_{i=0}^{n-1} \left\| \sum_{k=1}^{2} \left(2^{ki} \tilde{f}_k\left(\frac{x}{2^i}\right) - 2^{k(i+1)} \tilde{f}_k\left(\frac{x}{2^{i+1}}\right) \right) \right\| + \sum_{i=0}^{n-1} \left\| \sum_{k=3}^{5} \left(\frac{\tilde{f}_k(2^i x)}{2^{ki}} - \frac{\tilde{f}_k(2^{i+1}x)}{2^{k(i+1)}} \right) \right\| \\
&\le \sum_{i=0}^{n-1} \left(\frac{2^i}{5040} - \frac{4^i}{2688} \right) \left\| \Gamma \tilde{f}\left(\frac{x}{2^{i+1}}\right) \right\| \\
&\quad + \sum_{i=0}^{n-1} \left(\frac{1}{36864 \cdot 8^i} - \frac{1}{344064 \cdot 16^i} + \frac{1}{10321920 \cdot 32^i} \right) \|\Gamma \tilde{f}(2^i x)\| \\
&\le \frac{1}{2688} \sum_{i=0}^{n-1} 4^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{36864} \sum_{i=0}^{n-1} \frac{\Phi(2^i x)}{8^i}
\end{aligned}$$

for all $x \in V \setminus \{0\}$ by (2.2), (2.3), (2.4), (2.5), and (2.6), which follows (2.21) as $n \to \infty$.

(4) Let φ satisfy the condition(2.15), then F_{-1}, F_{-2}, F_{-3}, F_4, and F_5 are defined by (2.27). Putting a general quintic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_{-3}(x) + +F_4(x) + F_5(x)$$

for all $x \in V$. We have the inequality

$$\begin{aligned}
&\left\| \tilde{f}(x) - \sum_{k=1}^{3} 2^{kn} \tilde{f}_k\left(\frac{x}{2^n}\right) - \sum_{k=4}^{5} \frac{\tilde{f}_k(2^n x)}{2^{kn}} \right\| \\
&\le \sum_{i=0}^{n-1} \left\| \sum_{k=1}^{3} \left(2^{ki} \tilde{f}_k\left(\frac{x}{2^i}\right) - 2^{k(i+1)} \tilde{f}_k\left(\frac{x}{2^{i+1}}\right)\right) \right\| + \sum_{i=0}^{n-1} \left\| \sum_{k=4}^{5} \left(\frac{\tilde{f}_k(2^i x)}{2^{ki}} - \frac{\tilde{f}_k(2^{i+1}x)}{2^{k(i+1)}} \right) \right\| \\
&\le \sum_{i=0}^{n-1} \left(\frac{2^i}{5040} - \frac{4^i}{2688} + \frac{8^i}{4608} \right) \left\| \Gamma \tilde{f}\left(\frac{x}{2^{i+1}}\right) \right\| \\
&\quad + \sum_{i=0}^{n-1} \left(\frac{1}{344064 \cdot 16^i} - \frac{1}{10321920 \cdot 32^i} \right) \| \Gamma \tilde{f}(2^i x) \| \\
&\le \frac{1}{4608} \sum_{i=0}^{n-1} 8^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{344064} \sum_{i=0}^{n-1} \frac{\Phi(2^i x)}{16^i}
\end{aligned}$$

for all $x \in V \setminus \{0\}$ by (2.2), (2.3), (2.4), (2.5), and (2.6), which follows (2.22) as $n \to \infty$.

(5) Let φ satisfy the condition (2.16), then F_{-1}, F_{-2}, F_{-3}, F_{-4}, and F_5 are defined by (2.27). Putting a general quintic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_{-3}(x) + F_{-4}(x) + F_5(x)$$

for all $x \in V$. We have the inequality

$$\begin{aligned}
&\left\| \tilde{f}(x) - \sum_{k=1}^{4} 2^{kn} \tilde{f}_k\left(\frac{x}{2^n}\right) - \frac{\tilde{f}_5(2^n x)}{2^{5n}} \right\| \\
&\le \sum_{i=0}^{n-1} \left\| \sum_{k=1}^{4} \left(2^{ki} \tilde{f}_k\left(\frac{x}{2^i}\right) - 2^{k(i+1)} \tilde{f}_k\left(\frac{x}{2^{i+1}}\right)\right) \right\| + \sum_{i=0}^{n-1} \left\| \frac{\tilde{f}_5(2^i x)}{2^{5i}} - \frac{\tilde{f}_5(2^{i+1}x)}{2^{5(i+1)}} \right\| \\
&\le \sum_{i=0}^{n-1} \left\| \left(\frac{2^i}{5040} - \frac{4^i}{2688} + \frac{8^i}{4608} - \frac{16^i}{21504} \right) \Gamma \tilde{f}\left(\frac{x}{2^{i+1}}\right) \right\| + \sum_{i=0}^{n-1} \frac{1}{10321920 \cdot 32^i} \| \Gamma \tilde{f}(2^i x) \| \\
&\le \frac{1}{21504} \sum_{i=0}^{n-1} 16^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{10321920} \sum_{i=0}^{n-1} \frac{\Phi(2^i x)}{32^i}
\end{aligned}$$

for all $x \in V \setminus \{0\}$ by (2.2), (2.3), (2.4), (2.5), and (2.6), which follows (2.23) as $n \to \infty$.

(6) Let φ satisfy the condition (2.17), then F_{-1}, F_{-2}, F_{-3}, F_{-4}, and F_{-5} are defined by (2.27). Putting a general quintic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_{-3}(x) + F_{-4}(x) + F_{-5}(x)$$

for all $x \in V$. We have the inequality

$$\begin{aligned}
&\left\| \tilde{f}(x) - \sum_{k=1}^{5} 2^{kn} \tilde{f}_k\left(\frac{x}{2^n}\right) \right\| \le \sum_{i=0}^{n-1} \left\| \sum_{k=1}^{5} \left(2^{ki} \tilde{f}_k\left(\frac{x}{2^i}\right) - 2^{k(i+1)} \tilde{f}_k\left(\frac{x}{2^{i+1}}\right)\right) \right\| \\
&\le \sum_{i=0}^{n-1} \left\| \left(\frac{2^i}{5040} - \frac{4^i}{2688} + \frac{8^i}{4608} - \frac{16^i}{21504} + \frac{32^i}{322560} \right) \Gamma \tilde{f}\left(\frac{x}{2^{i+1}}\right) \right\| \\
&\le \sum_{i=4}^{n-1} \left(\frac{2^i}{5040} - \frac{4^i}{2688} + \frac{8^i}{4608} - \frac{16^i}{21504} + \frac{32^i}{322560} \right) \left\| \Gamma \tilde{f}\left(\frac{x}{2^{i+1}}\right) \right\| \\
&\le \frac{1}{322560} \sum_{i=4}^{n-1} 32^i \Phi\left(\frac{x}{2^{i+1}}\right)
\end{aligned}$$

for all $x \in V\backslash\{0\}$ by (2.2), (2.3), (2.4), (2.5), and (2.6), since $\frac{2^i}{5040} - \frac{4^i}{2688} + \frac{8^i}{4608} - \frac{16^i}{21504} + \frac{32^i}{322560} = 0$ when $i \in \{0,1,2,3\}$, which follows (2.24) as $n \to \infty$.

Moreover, by the definition, we easily get

$$F_{\delta k}(2x) = 2^k F_{\delta k}(x)$$

and $\overset{6}{\underset{y}{\Delta}} F_{\delta k}(x) = 0$ for all $x, y \in V$. According to Lemma 2.4, F is the unique general quintic mapping. □

The stability results for the functional equation (1.1) proved by Lee and Jung (2023) and Jin and Lee (2021) only deal with the conditions (2.12) and (2.17) of Theorem 2.4. Compare the following concise theorem obtained from Theorem 2.4 with Theorem 1.1 obtained by Lee (2019b).

Theorem 2.6. *Let θ be a positive real constant and p a real number such that $p \neq 1,2,3,4,5$. If $f : X \to Y$ satisfies the inequality*

$$\left\| \overset{6}{\underset{y}{\Delta}} f(x) \right\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X\backslash\{0\}$, then there exists a unique general quintic mapping F such that

$$\begin{aligned}
\|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{5040(2-2^p)} && \text{for} \quad p < 1, \\
\|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{5040(2^p-2)} + \frac{M\theta\|x\|^p}{2688(4-2^p)} && \text{for} \quad 1 < p < 2, \\
\|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{2688(2^p-4)} + \frac{M\theta\|x\|^p}{4608(8-2^p)} && \text{for} \quad 2 < p < 3, \\
\|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{4608(2^p-8)} + \frac{M\theta\|x\|^p}{21504(16-2^p)} && \text{for} \quad 3 < p < 4, \\
\|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{21504(2^p-16)} + \frac{M\theta\|x\|^p}{322560(32-2^p)} && \text{for} \quad 4 < p < 5, \\
\|\tilde{f}(x) - F(x)\| &\leq \frac{1024M\theta\|x\|^p}{315 \cdot 16^p(2^p-32)} && \text{for} \quad 5 < p
\end{aligned}$$

for all $x \in X\backslash\{0\}$, where

$$\begin{aligned}
M :=& 21 \cdot 24^p + 1680 \cdot 12^p + 57 \cdot 8^p + 8400 \cdot 6^p \\
&+ 1834 \cdot 4^p + 5376 \cdot 3^p + 19040 \cdot 2^p + 58624.
\end{aligned}$$

3 Conclusions

In this paper, we investigate the generalized stability of the general quintic functional equation (1.1). Precisely, if $f : V \to Y$ is a mapping such that $\left\| \Delta_y^6 f(x) \right\| \leq \varphi(x,y)$ for all $x, y \in V\backslash\{0\}$, where $\varphi : (V\backslash\{0\})^2 \to [0,\infty)$ holds the conditions (2.12), (2.13), (2.14), (2.15), (2.16), or (2.17), then there exists a unique general quintic mapping F such that the difference $\|\tilde{f}(x) - F(x)\|$ satisfies the conditions (2.19), (2.20), (2.21), (2.22), (2.23), or (2.24), respectively.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

Competing Interests

Authors have declared that no competing interests exist.

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